MANAGING MULTIPLE COMMONS: STRATEGY-PROOFNESS AND MIN-PRICE WALRAS

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Abstract. There are several locations, each of which is endowed with a resource that is specific to that location. Examples include coastal fisheries, oil fields, etc. Each agent will go to a single location and harvest some of the resource there. Several agents may go to each location. Selling the commons for money is not desirable, either because agents have equal right to use the resources or because control of the commons would give unacceptable market power to its owner. Thus we will assign harvesting rights based on preferences alone, though the model can be extended to accommodate private endowments of money. We find the best allocation rule in the class of rules that are strategy-proof, anonymous, and that satisfy a weak continuity property. We also find an ascending mechanism, similar to an auction, that implements the rule. The rule is defined via a simulated price equilibrium, wherein agents buy their desired resource with tokens distributed by the social planner. Equilibria of this form are not unique as full distribution of the resources is not required. However, equilibrium price vectors form a lower semi-lattice and thus there is a unique minimal price vector. The equilibria associated with the minimal price vector are called min-price Walrasian equilibria. These equilibria form an essentially single-valued correspondence, and this correspondence is the rule we characterize.

1. Introduction

We study the centralized administration of several commonly owned resources. Each agent in the population is equally entitled to consume the resources but may consume only one type. Examples may include coastal fisheries or oil fields. There are several coastal fisheries, but a boat can be in only one place at one time and there are only so many fish. Similarly, an oil company may have several drilling units that it may locate among a smaller number of fields. Each unit has private information about its cost per barrel of oil extracted, and these costs vary by field and quantity. The firm needs to distribute its target level of extraction amongst these units. In general, we study an economy with several common areas and potentially many agents who wish to use them. Each agent may go to only one common area. Agents have preferences over the areas and the amount of some underlying resource that they enjoy there. The presence of many agents consuming at a desirable area will make it less desirable by competing for resources.

We avoid debate on the nature of common ownership, but we assume that it makes auctioning of the resources undesirable or politically infeasible. For example, citizens may not accept being excluded from a fishery they consider a national resource. A flat license fee may be acceptable whereas price discrimination may not. Moreover, the greatest social benefit may not be extracted from the resources if they are auctioned to the highest bidder or bidders. The bidders may be motivated by acquiring market power, and as a result, their reported valuations may include expected monopoly rents. Even if market power is not a consideration, a winning bidder is likely to suffer from the so-called "winner's curse." For a resource such as a fishery, this could be dangerous: if a winning firm finds itself unable to break-even, it may resort to more destructive fishing practices. The authority nonetheless wishes to distribute resources amongst users in the most efficient way possible. To do so, it must elicit the preferences of the agents. Thus, our primary aim is the study of rules that elicit truthful reporting as a weakly dominant strategy.

One simple solution is to impose an exogenous order on agents and allow them to take their desired bundle in turns, each potentially leaving some resources for the next. We confront a choice: which order should we use? If we are unwilling, however, to discriminate based on agents' ability to pay, we ought to have a compelling reason to discriminate based on their identity (a prior claim to certain resources, for example). We have in mind applications where there is no such reason, so we impose an *anonymity* condition.[1] Under this condition there is unfortunately no *Pareto efficient* (henceforth simply *efficient*) rule that elicits true preferences as a weakly dominant strategy. This is not surprising. The Vickrey-Clarke-Groves rules are not budget balanced and therefore not *Pareto efficient*.[2] We can, however, identify and characterize a constrained optimal rule for this model and we find that its deviation from *Pareto efficiency* is not great. In particular, we characterize the best rule in the class satisfying *strategy-proofness*, *anonymity in welfare terms*, and a weak continuity axiom. We also find that if n people consume a given resource then at most a $1/n+1$ fraction of it will be left undistributed.

We introduce the formal model in Section 2. In Section 3 we introduce our solution concept, which is a form of simulated price equilibrium, and derive from it a rule. We analyze the properties of the rule and provide some comparative statics. Section 4 gives the characterization and section 5 the auction mechanism. The appendices contain proofs of lower pedagogical value. First, we describe a potential application of the model and we provide a brief review of the literature and it's relation to the present work.

1.1. **Application: Coastal Fisheries.** The coastal fisheries of the United States are regulated by the federal government's Department of Commerce. The fisheries are divided into 9 regions, the authority over each being delegated to a council for that region. The mandate of the councils includes, among other things, "ensuring the equitable allocation of fishing privileges, preventing excessive accumulation of quota [consolidation], using fishery resources efficiently, . . ., and considering the importance of fishery resources to fishing communities" (GAO (2002)). One method for achieving this, which has been used in Alaska, the Mid-Atlantic, and the South Atlantic, is the Individual Fishing Quota (IFQ) system. For example, in Alaska, halibut and sablefish are administered by IFQ's. The fishery of each species is divided into zones, 8 for halibut and 6 for sablefish. For each species and each of its zones, the council sets a maximum allowable catch, based on scientific assessment of the current health of the stock. The allowable catch is then distributed among eligible entities.

[1] Kolm (2002) argues that symmetric treatment of the agents is the most "rational" approach. To the layman, a choice is rational if it has a reason. If there is no reason to choose one ordering of agents over another, then any order-based rule is, in this sense, irrational. See Kolm (2002) for a complete argument.

[2] If the agents supply the goods and money, there is no efficient, anonymous redistribution that does not sometimes burn money. Considering a model with an auctioneer, whose preferences is known, side-steps this problem.

Of the eight halibut zones, two have the preponderance of the allowable catch.[3] Thus, I consolidated some of the less significant zones that were next to each other on the coast and arrived at a new partition of 4 zones. In practice, 80% of the fishing entities hold at most one halibut license under my partition. The distribution of sablefish licenses is more diffuse and over 70% of entities holding sablefish licenses also hold at least one halibut license. License ownership, however, is not necessarily an accurate indicator of actual use, nor of the underlying technology; only 17% of fishing journeys result in catching appreciable numbers of both species. While halibut and sablefish are generally found in the same area, their great difference in size–the average halibut is 40 lbs while the average sablefish 11–and the differences in the depth at which they are found make specialization likely. In sum, the data available do not allow us to assert a perfect fit between our model and the real world, but the following specification is suggested: 5 common resources, 4 of which have halibut and one having sablefish. We emphasized that the resources need not be the same, and they are not in this example. The four zones of halibut may also be less similar than they first appear: the zones together cover a very large area, with diverse weather patterns and hazards such as ice or rough seas. Catching halibut in a given zone is not necessarily equivalent to halibut in another, even though the fish is the same.

The solution we provide, as a benchmark, gives all agents equal opportunity to harvest. Thus, any consolidation that results from its application is purely an expression of preferences and, therefore, is beneficial. Such radical equality is probably not appropriate for applications, but it provides a theoretical starting point and confirmation that the government's objectives can be achieved with minimal loss of efficiency.

1.2. **Relation to the Literature.** The "tragedy of the commons," a phrase coined by Hardin (1968), is the well known propensity of resources held in common to be overused. Hardin's essay was a non-formal appeal to control population growth. Economists followed with theoretical work, demonstrating that, at Nash equilibrium, a commonly owned productive resource will be overused (Moulin and Watts (1996)). Roemer and Silvestre (1993) proposed a solution, supported by normative axioms. Theirs is an extension of the Walrasian

[3]The data used in the following analysis can be found at http://alaskafisheries.noaa.gov/ram/ifqreports.htm. License holding is from the 2013 "Current Quota Share with Holders and QS units" dataset. Fish landings are taken from 2013 "Allocations and Landings" report.

solution, which they applied to a rich model with private goods, private production, and one publicly owned production technology that requires private inputs.

An alternative, and more simple solution to the problem has long been known to economists: eliminate the commons by allocating property rights. In some contexts, such a transaction may be repugnant, but we consider the problem of a commons that can appropriately be exploited for private benefits. The complication arises then from the fact that the common nature of the resource implies some prior right is held by all members of a population. In our leading example, this population is the community of fisherman. Auctioning the right to fish may exclude members of this population, in violation of the principle. Thus, our problem is one of allocating usage rights when agents are supposed to have equal opportunity to receive them. While this problem has not be studied, a mathematically similar problem has been studied extensively: the problem of allocating a finite set of objects and a quantity of money. This problem can be viewed as a special case of the model we propose below, if the appropriate adjustments are made. In what follows, the phrase “objects-and-money” will refer to the class of models with the following characteristics: There is a finite set of indivisible objects and a single divisible commodity. The divisible commodity may or may not come in bounded supply. Agents preferences are monotone in the divisible commodity. Agents' welfare may be increased by the consumption of a single object but is always unchanged by the consumption of a second, and thus we may assume without loss that agents consume at most one object. Only one agent may consume a commodity; there is no sharing.

At least as early as 1960, in his book “The Theory of Linear Economic Models,” Gale (1960) considered an embryonic version of the objects-and-money model. For simple preferences, he applied integer programming techniques to find an efficient allocation when transfers are unbounded. Leonard (1983) applied these techniques to further study the space of prices that support an efficient allocation. He found that there is a minimal such price vector and that these correspond to VCG payments. Therefore, the rule that, for each economy, prices goods at the minimal supporting price, is strategy-proof. Demange and Gale (1985) discovered that both the lattice structure and the strategy-proofness of this minimal-price rule continue to hold even when preferences are not quasilinear (utility is not perfectly transferable). Note however that in all of these environments, as in the typical applications of the Groves scheme, material balance in money is not enforced.

In classical economic environments, with a convex consumption space, if the feasibility constraint requires material balance in all dimensions, Pareto efficiency and strategy-proofness together imply poor equity properties (Serizawa (2002); Serizawa and Weymark (2003); Hurwicz (1972)). The same is true for objects-and-money. When negative consumption of money is prohibited, and when there are just two agents, the only *strategy-proof* and *efficient* rules are dictatorial (Schummer (2000)). For more than two agents, Schummer (2000) and a subsequent paper by Svensson and Larsson (2002) study the additional axiom of *non-bossiness*, which requires that no agent can effect a change in the assignment of other agents without also changing his own. *Strategy-proofness* and *non-bossiness* together obviate the role of money in the model: such a rule fixes, in advance, the quantity of money associated with each object.

Without insisting on incentive compatibility, objects-and-money models, with exact material balance, are remarkable in the equity properties they admit. In the classical model, the existence of an *efficient* and *envy-free* allocation is not guaranteed when preferences are not convex (Varian (1974)). In this model, *envy-freedom* implies *efficiency*, and *envy-free* allocations always exist (Svensson (1983)). Svensson further demonstrated the equivalence between *envy-freeness* and equal-income Walrasian equilibria, which is one motivation for our application of a Walrasian-type solution concept to our model. Even in the presence of consumption externalities, there exists an *efficient* and *envy-free* allocation (Gale (1984), and Velez (2014)).

Similarly, in the classical model, there is tension between *efficiency* and *resource monotonicity*, the requirement that no agent be harmed by an increase in social endowment (Moulin and Thomson (1988)). This tension is partially relieved here: there are *envy-free* solutions that are welfare monotonic in money, though not necessarily in the addition of more objects (Alkan et al. (1991)).

Not all of the news is good. Tadenuma and Thomson (1991) show that no selection from the *envy-free* and *efficient* correspondence satisfies *consistency*. This is an indication of the interdependence between agents induced by envy-freedom, interdependence that will continue to hold in our equilibrium concept. We should not expect our rule to be consistent, and in fact it is not. Moreover, the close relationship between *consistency* and *non-bossiness* leads us to be pessimistic as well about the latter. Our rule will be *bossy*.

Thus if we want incentive compatibility and any sort of equity we are forced to accept disposal of some goods. Here the application of minimal-price Walrasian rules has been fruitful. Two cases are focal: (i) when there is a quantity of money to be divided along with the objects and (ii) when the agents are to pay for the goods they acquire. In the second case, we may regain material balance of money by assuming an auctioneer, whose preferences are known, claims the money collected. In the first case, for quasilinear preferences, Svensson (2009) provides the solution. To each object associate a quantity of money such that the sum of these quantities is the social endowment of money. These quantities serve as an upper bound; no allocation will associate to any object more than its designated quantity of money. Now using the lattice property of *envy-free* allocations, which is akin to the lattice structure of Walrasian prices, find the Pareto best *envy-free* allocation among those that respect the money bound. Typically, such an allocation will associate to some object strictly less than its money bound, and therefore, money will be wasted. Svensson tells us that we must accept this loss: among the "regular" rules (a mild condition), those that are *envy-free* and *weakly group-strategy-proof* are of this form. In the second case, the money is provided by the agents and its quantity is not set in advance. It is natural, though, to assume that agents who are not assigned an object are also not given any more money than they brought, a condition called *no subsidy for losers*. Moreover, since we may imagine an auctioneer, it is useful to include him in the calculation of *efficiency*, though we must remember that his preferences need not be elicited. Morimoto and Serizawa (2012) then tell us that *efficiency*, *individual rationality*, *strategy-proofness* and *no subsidy for losers* characterize the min-price Walrasian rule in this environment. This is a generalization of Holmström's (1977) result for quasilinear preferences.

The models in the previous paragraph bound an agent's consumption of money on at most one side. In reality, given price controls, consumption of money may be bounded on both sides. This case is studied by Andersson and Svensson (2014). With the consumption space so constrained, the usual notion of equilibrium often fails to exist: prices do not have sufficient power to influence demand. But by dropping *envy-freedom*, which is implicitly guaranteed in case (ii) above because all agents have equal access to credit, they can define a type of equal income Walrasian equilibrium that allows for priority based rationing of the goods. Their equilibrium always exists, and on almost all profiles, there is a unique minimal equilibrium price. On the sub-domain for which it is defined, the rule that always selects the minimal price and allocates an associated equilibrium is *strategy-proof*.

All of the work heretofore mentioned assumes no object can be shared. This is not the case for Abizada (2013), whose model is the most similar to ours. In his model, each object is associated with a fixed quantity of money and a priority ranking over agents. Each agent may consume only one object, but each object is non-rival: many agents may consume it without diminishing the enjoyment of others. Agents have quasilinear preferences. Abizada finds a strategy proof rule that satisfies *no-justified-envy*: if i prefers the assignment of j, then j has higher priority for his object than i has. The rule is found via a modification of the Deferred Acceptance algorithm and so he does not obtain the structural results we require, nor is it clear if this approach could be directly applied to this problem, as we will insist upon *anonymity*. Nevertheless, Abizada's paper and the one by Andersson and Svensson (2014) suggest that dropping anonymity is worthwhile.

2. Model

There are a finite set S of sites and a finite set N of agents. Generic sites are denoted s and t, and generic agents are denoted i and j. At each site $s \in S$, there is a social endowment of $e_s \in \mathbb{R}_{++}$ units of a divisible commodity specific to that site. We emphasize that the commodity associated with each site may be different and, therefore, the sum $\sum e_s$ need not be meaningful. We assume for simplicity that the number of agents allowed at each site is $|N|$, but relaxing this doesn't change most of the results. Occasionally it is useful to normalize the endowment so that for each site $s \in S$, $e_s = 1$. In this case we write $e = \mathbf{1}$.

Each agent is assigned to a single site and given a non-negative quantity of that site's commodity. Thus a typical consumption bundle is a pair $(x_i, s_i) \in \mathbb{R}_+ \times S$. Preferences, however, are defined over $\mathbb{R} \times S$. A typical preference relation is denoted R_i with symmetric and antisymmetric parts I_i and P_i, respectively. A preference relation R_i is **increasing** if for each pair $x_i,\ y_i \in \mathbb{R}$ with $x_i > y_i$ and for each site $s \in S$, $(x_i, s_i)\ P_i\ (y_i, s_i)$. The set of continuous, increasing preferences is denoted $\mathcal{R}$. If X is a set of bundles, we write $(x_i, s_i)\ R_i\ X$ to mean that for each $(y_i, t_i) \in X$, $(x_i, s_i)\ R_i\ (y_i, t_i)$.

A function $\alpha : N \to S$ is called a **site assignment**. An **allocation** is a pair $(x, \alpha) \in \mathbb{R}_+^N \times S^N$, i's bundle being given by $(x_i, \alpha(i))$. Agents are self-centered and so we extend their preferences to the space of allocations in the usual way: $(x, \alpha)\ R_i\ (y, \beta)$ if and only if $(x_i, \alpha(i))\ R_i\ (y_i, \beta(i))$. An allocation (x, α) is **feasible** if each site distributes no more commodity than its endowment. Formally, for each site s

$$\sum_{i \in \alpha^{-1}(s)} x_i \leq e_s. \tag{1}$$

Let $\mathcal{D}$ denote a generic sub-domain of $\mathcal{R}^N$. A mapping $\varphi : \mathcal{D} \to 2^{\mathbb{R}^N \times S^N}$ is **single-valued** if, for each $R \in \mathcal{D}$, $\varphi(R)$ is a singleton. A mapping φ is **essentially single-valued** if, for each $R \in \mathcal{D}$, $\varphi(R)$ is a singleton *in welfare*: for each $R \in \mathcal{D}$, each agent i is indifferent between all the allocations in $\varphi(R)$. A mapping φ is **non-empty-valued** if for each $R \in \mathcal{D}$, $\varphi(R) \neq \emptyset$. Finally, a **rule** is a *non-empty-valued* mapping $\Phi : \mathcal{R} \rightrightarrows \mathbb{R}^N \times S^N$ that is *essentially single-valued.*

Given a rule Φ, a **selection** from Φ is a function $\hat{\varphi} : \mathcal{D} \to \mathbb{R}^N \times S$ such that for each $R \in \mathcal{D}$, $\hat{\varphi}(R) \in \Phi(R)$. We write $\varphi \in \Phi$ to indicate that φ is a selection from Φ. Since rules are non-empty-valued, we often conflate the function $\hat{\varphi} : \mathcal{D} \to \mathbb{R}^N \times S^N$ with the single-valued rule $\varphi : \mathcal{D} \rightrightarrows \mathbb{R}^N \times S^N$ defined by the equation $\varphi(R) = \{\hat{\varphi}(R)\}$. The properties of rules that we study are more easily understood when defined for their selections. Therefore, all properties are defined on single-valued rules and extended as follows: Given a property $\mathbf{P}$ defined for single-valued rules, the generic rule Φ satisfies $\mathbf{P}$ only if each selection $\varphi \in \Phi$ satisfies $\mathbf{P}$.

In many environments, agents have the right to abstain from participation and thereby avoid all consequences, positive or negative, of the rule used. Our model accommodates this easily by setting $e_\emptyset = 0$. Recall that the capacity of site $\emptyset$ is $|N|$, the size of the population. If $e_\emptyset = 0$, then consuming at $\emptyset$ implies consuming the bundle $(0, \emptyset)$. Since we may feasibly allocate this bundle to all agents simultaneously, we may view it as the outside option.

3. Price Equilibrium.

We augment the economy with a special divisible good that agents may use to purchase their bundles. This good is a tool for realizing an allocation and, afterward, all of the good is collected and destroyed. We call this good the *numeraire.*

3.1. **Definition.** The manager of the commons endows each agent with w units of the numeraire. Given price vector $p \in \overline{\mathbb{R}}_{+}^{S}$, all agents face the budget set[4]

$$\{(x_0, s_0) \in \mathbb{R} \times S : \; p_{s_0} x_0 \leq w\}.$$

We define the **demand correspondence $\boldsymbol{D}$** for each preference relation R_i and price vector p, by

$$D\left(R_i, p\right) := \{(x_0, s_0) : (y_0, t_0) \; P_i \; (x_0, s_0) \implies p_{t_0} y_0 > w\}.$$

Denote by D_S the projection of the demand correspondence onto the set of sites. The **site demand $\boldsymbol{D_S(R_i, p)}$** indicates which site assignments are considered best by preference relation R_i at prices p.

The solution concept we study is a form of price equilibrium in which we *do not* require exact material balance. An implication of Theorem 1 below is that anonymous pricing is incompatible with the allocation of all resources.

Equilibrium: A price vector p and an allocation (x, α), satisfying the following conditions:

(1) (x, α) is feasible;
(2) $\forall i \in N$, $(x_i, \alpha(i)) \in D(R_i, p)$;
(3) If $\alpha^{-1}(s) = \emptyset$ then $e_s p_s = w$.

If price vector p admits an equilibrium allocation then we call it an **equilibrium price vector**. The set of equilibrium price vectors for profile R is $\mathbb{P}(R)$. For each economy, there is at least one equilibrium, given by the price vector $p = |N|\,(w, w, \ldots, w)$. Faced with these prices, each agent will choose to spend his entire endowment w of numeraire to get $^{1}\!/\!_{|N|}$ of some commodity. It is clear that an equilibrium may be constructed from such choices. Thus, for each $R \in \mathcal{R}^N$, $\mathbb{P}(R)$ is non-empty.

3.2. **Minimal Prices.** Given prices p, let $\tilde{c}_s(p)$ denote the integer satisfying

$$w\tilde{c}_s(p) \leq p_s e_s < w\left(\tilde{c}_s(p) + 1\right).$$

The **implied capacity of site** s **given prices** p, defined as $c_s(p) := \min\{c_s, \tilde{c}_s(p)\}$, is the largest number of agents that can be assigned to s under price p_s. Note that if $p_s e_s < w$ then $c_s(p) = 0$.

[4]$\overline{\mathbb{R}}$ denotes the affinely extended real line: $[-\infty, +\infty]$.

Given an equilibrium price vector p, all agents are indifferent between all of the equilibrium allocations it admits. If p and p' are equilibrium price vectors and $p' > p$, then all agents prefer all of the p equilibria to all of the p' equilibria. Thus, social welfare is decreasing in the order $>$ on $\mathbb{P}(R)$ and non-increasing in $\geq$. Given two elements, p and $p' \in \mathbb{R}^S$, let $p \vee p'$ and $p \wedge p'$ be the component-wise maximum and minimum, respectively, of p and p'. That is, for each s,

$$(p \vee p')_s := \max\{p_s, p'_s\} \quad \text{and} \quad (p \wedge p')_s := \min\{p_s, p'_s\}.$$

If $A \subset \mathbb{R}^k$ and if for each pair $\{p, p'\} \subseteq A$, $p \wedge p' \in A$, then we say the pair $(A, \wedge)$ is a **lower semi-lattice**.

Theorem 1. *For each $R \in \mathcal{R}^N$, $(\mathbb{P}(R), \wedge)$ is a lower semi-lattice.*

Proof. For each $s \in S$, each $\hat{p} \in \mathbb{R}^S_+$, let $\mathcal{I}_s(\hat{p}) := \{N' \subseteq N : \ |N'| \leq c_s(\hat{p}),\ \forall i \in N,\ s \in D_S(R_i, \hat{p})\}$. Clearly $\mathcal{M}_s(p \wedge q) := (N, \mathcal{I}_s(p \wedge q))$ is a matroid, whose rank function we denote $\rho_s(\cdot)$.[5] Note that $p \wedge q$ admits an equilibrium if and only if the matroids $(\mathcal{M}_s(p \wedge q))_{s \in S}$ are partitionable.

Let $S_p := \{s \in S : \ p_s = (p \wedge q)_s\}$ and $S_q := \{s \in S : \ q_s = (p \wedge q)_s\}$. Let $N_p := \{i \in N : D(R_i, p \wedge q) \cap S_q = \emptyset\}$ and $N_q := \{i \in N : \ D(R_i, p \wedge q) \cap S_p = \emptyset\}$. For each $N' \subseteq N$, denote by $\mathcal{M}_s(\hat{p}) \setminus N'$ the matroid $\mathcal{M}_s(\hat{p})$ delete N'. We now show that if $(\mathcal{M}_s(p \wedge q) \setminus N_p)_{s \in S_q}$ and $(\mathcal{M}_s(p \wedge q) \setminus N_q)_{s \in S_p}$ are both partitionable then $(\mathcal{M}_s(p \wedge q))_{s \in S}$ is partitionable. For each $N' \subseteq N$,

$$\begin{aligned} |N' \setminus N_p| + |N' \cap N_p| &\leq \sum_{s \in S_q} \rho_s(N' \setminus N_p) + \sum_{s \in S_p} \rho_s(N' \cap N_p) \\ &= \sum_{s \in S_q} \rho_s(N' \setminus N_p) + \sum_{s \in S \setminus S_q} \rho_s(N' \cap N_p) \leq \sum_{s \in S} \rho_s(N'). \end{aligned}$$

The first inequality is by Edmond's matroid partition theorem. The second equality is because, for each $s \in S_q$, $\rho(N_p) = 0$. The third, and final, inequality is by monotonicity of the rank function. The resulting inequality, $|N'| \leq \sum_{s \in S} \rho_s(N')$, implies via Edmond's theorem that $(\mathcal{M}_s(p \wedge q))_{s \in S}$ is partitionable, proving the claim.

[5]See Vohra (2004) for a short reference on matroid theory and Edmond's theorem.

For $i \in N \setminus N_p$, there exists $s \in D_S(R_i, p \wedge q)$ such that $q_s = (p \wedge q)_s$. Let $t \in D_S(R_i, q)$. Then

$$\left(\frac{w}{(p \wedge q)_t}, t\right) R_i \left(\frac{w}{q_t}, t\right) R_i \left(\frac{w}{q_s}, s\right) I_i \left(\frac{w}{(p \wedge q)_s}, s\right), \tag{2}$$

and so $t \in D_S(R_i, p \wedge q)$. On the other hand, if $r \in S$ satisfies $p_r < q_r$, then

$$\left(\frac{w}{q_s}, s\right) I_i \left(\frac{w}{(p \wedge q)_s}, s\right) R_i \left(\frac{w}{(p \wedge q)_r}, r\right) P_i \left(\frac{w}{q_r}, r\right) \tag{3}$$

and therefore $r \notin D_S(R_i, q)$. For each $s \in S$, let $\hat{\rho}_s$ be the rank function of $\mathcal{M}_s(q)$. Let $N' \subseteq N \setminus N_p$ and $s \in S_q$. Line 2, and the fact that $c_s(q) = c_s(p \wedge q)$, imply that $\rho_s(N') \geq \hat{\rho}_s(N')$. Line 3 implies that for each $t \in S \setminus S_q$, $\hat{\rho}_t(N') = 0$.

If $(\mathcal{M}_s(p \wedge q))_{s \in S}$ is not partitionable, one of the deleted matroid families is not partitionable. Assume without loss of generality that $(\mathcal{M}_s(p \wedge q) \setminus N_p)_{s \in S_q}$ is not partitionable. Therefore, by Edmond's theorem, there exists $N' \subseteq N \setminus N_p$ such that

$$|N'| \;>\; \sum_{s \in S_q} \rho_s(N') \geq \sum_{s \in S_q} \hat{\rho}_s(N') = \sum_{s \in S_q} \hat{\rho}_s(N') + \sum_{s \in S \setminus S_q} \hat{\rho}_s(N') = \sum_{s \in S} \hat{\rho}(N').$$

contradicting the assumption that $(\mathcal{M}_s(q))_{s \in S}$ is partitionable. ■

Lattice structures have been observed in both the objects-and-money allocation problem and in the matching problem with money. For objects-and-money, Demange and Gale's (1985) model of one-to-one matching is readily adapted to the case where one side is not agents but objects. The Decomposition Lemma introduced by Demange and Gale, and used by several papers in the literature, cannot be applied to the same effect here: we cannot guarantee that the agents indifferent between two different equilibria are consuming at sites whose prices are the same in both. Moreover, we cannot introduce "dummy agents" as we do not know in advance how many we will need. Thus, our Theorem 1 is not implied by previous work.

Note that by continuity of preferences, $\mathbb{P}(R)$ is closed. If $\mathbb{P}(R)$ is non-empty, then it is bounded below by 0. Therefore, by Lemma 1, $\mathbb{P}(R)$ has a unique smallest element $p^*(R)$. If $p^*(R)$ is defined, let $\mathcal{A}^*(R)$ be the set of site assignments α such that for some $x \in \mathbb{R}^N$, (p, x, α) is an equilibrium.

The Min-Price Rules: Fix a domain $\mathcal{D}$ such that for each $R \in \mathcal{D}$, $\mathbb{P}(R)$ is non-empty. Then p^* is defined on $\mathcal{D}$. Let (x, α) and (y, β) be two equilibria, both supported by price $p^*(R)$. Then (x, α) and (y, β) are Pareto indifferent for profile R. Thus the mapping $F^* : \mathcal{D}^N \rightrightarrows \mathbb{R}^S_+ \times S^N$ given for each profile R by

$$F^*(R) = \{|p^*(R), \alpha| : \alpha \in \mathcal{A}^*(R)\}$$

is a rule. We call F^* the *min-price rule* on domain $\mathcal{D}$. With mild abuse of notation, if an arbitrary domain $\tilde{\mathcal{D}}$ admits a min-price rule, then we denote that rule by F^*.

3.3. **Properties of Min-price Rules.**

3.3.1. *Welfare properties and comparative statics.* The outcome of a min-price rule is not always *Pareto efficient*. It is efficient in a limited sense: given an allocation $(p, \alpha) \in F^*(R)$ and a permutation $\pi \in N^N$, for each agent i, $(x_i, \alpha(i))\ R_i\ \big(x_{\pi(i)}, (\alpha \circ \pi)(i)\big)$. Welfare cannot be improved by having agents exchange bundles. Unfortunately, for many preference profiles, there will be no allocations in F^* that distribute all of the social endowment. We can calculate the quantity of undistributed resources and we argue that it is small. Moreover, this lack of efficiency is not novel among anonymous social choice rules. Recall that Groves mechanisms are generically not budget balanced. The Vickrey rule for allocating objects is efficient only because of the residual claimant. If there is no such agent, then the Vickrey rule suffers the same problem as the min-price rules.

If the resource of a site is completely distributed, we say the site is **exhausted**. At any min-price equilibrium, at least one site is exhausted. What prevents all sites from being exhausted are the preference relations of those consuming at exhausted sites. To see why, beginning at a min-price equilibrium, suppose the price of a non-exhausted site t were lowered. This will cause a consumer, i, of an exhausted site, s, to demand t. But if he moves to t, to retain feasibility, the price of t will have to rise again and he will regret going there. The only way to achieve an equilibrium then is for the price of t to remain high and i to remain at s.

The tension discussed in the previous paragraph can be transmitted via chains of indifference, and these chains decide the equilibrium price list. Formally, let $(x, \alpha) \in F^*(R)$. If there is a statement of the form

$$\big(x_{i^1}, \alpha(i^1)\big)\ I_{i^1}\ \big(x_{i^2}, \alpha(i^2)\big)\ I_{i^2} \cdots I_{i^n}\ (x_i, \alpha(i)),$$

then site $\alpha(i)$ is **blocked via indifference by site** $\alpha(i^1)$. The following lemma shows that if a site is not exhausted, it is blocked via indifference by a site that is.

Lemma 1. *Let $R \in \mathcal{R}^N$ and $(x, \alpha) \in F^*(R)$ be supported by prices p. Assume that s is not exhausted at equilibrium (p, x, α). Thenk either $s \notin \cup_{i \in N} D_S(R_i, p)$ or s is blocked via indifference by a site t that is exhausted at (p, x, α).*

The lemma implies the existence, at any min-price equilibrium, of at least one exhausted site.

Only if $wc_s(p) = p_s e_s$ is it possible for s to be exhausted at p. We will refer to such sites as having **endowment-divisible value**. Given $p \in \mathbb{P}(R)$, say that $\alpha \in N^S$ is **balanced** if for each site s having endowment-divisible value, $|\alpha^{-1}(s)| \geq c_s(p) - 1$, and for every other site t, $|\alpha^{-1}(t)| = c_t(p)$. The following proposition is shown in the appendix.

Proposition 1. *For each profile R, there is a balanced site matching $\alpha \in \mathcal{A}^*(R)$.*

We use balanced assignments to calculate the undistributed resources. At a balanced, min-price allocation, the amount discarded of each resource is bounded above by ${}^{e_s}/_{k+1}$ if k people consume the resource.

Theorem 2. *For each profile R such that $F^*(R)$ is non-empty, there is an allocation $(x, \alpha) \in F^*(R)$ such that for each site s,*

$$\sum_{i \in \alpha^{-1}(s)} x_i \geq \left(\frac{|\alpha^{-1}(s)|}{|\alpha^{-1}(s)| + 1} \right) e_s.$$

Proof. Let $(x, \alpha) \in F^*(R)$ be such that α is balanced. For notational simplicity, let $p = p^*(R)$. If site s has endowment-divisible value at p, then balancedness of α gives $|\alpha^{-1}(s)|+1 \geq c_s(p)$ and endowment-divisibility gives $wc_s(p) = p_s e_s$. Thus, $|\alpha^{-1}(s)| + 1 \geq {}^{p_s e_s}/_{w}$. Since $(|\alpha^{-1}(s)|\, w)\, p_s^{-1} = \sum_{i \in \alpha^{-1}(s)} x_i$, dividing both sides by $|\alpha^{-1}(s)|$ gives

$$\frac{|\alpha^{-1}(s)| + 1}{|\alpha^{-1}(s)|} \geq \frac{e_s}{\sum_{i \in \alpha^{-1}(s)} x_i}.$$

If site s is not endowment-divisible, $|\alpha^{-1}(s)| = c_s(p)$ and so

$$\left|\alpha^{-1}(s)\right| + 1 = c_s(p) + 1 > \frac{p_s e_s}{w}.$$

Again dividing through by $|\alpha^{-1}(s)|$ gives the result. ∎

We now analyze the response of prices to changes in preferences. The result is a theorem that provides useful tools for comparative statics. We later use these tools to demonstrate the incentive properties of min-price rules.

For analytical precision, we confine ourselves to a class of preference transformations that represent an unambiguous strengthening or weakening of preference for a given site or set of sites. Let $R_i \in \mathcal{R}$ and $\hat{S} \in S$. Define $\boldsymbol{R_i^{\hat{S},d}}$ so that $R_i^{\hat{S},d}\Big|_{\mathbb{R}\times S\setminus\hat{S}} = R_i\Big|_{\mathbb{R}\times S\setminus\hat{S}}$ and for each $\hat{s} \in \hat{S}$ and each $t \in S \setminus \hat{S}$,

$$(x, \hat{s})\ R_i\ (y,t) \implies (x-d, \hat{s})\ R_i^{\hat{S},d}\ (y,t).$$

We say that $R_i^{\hat{S},d} R_i^{d,s}$ is a **site-translation through set** $\hat{S}$, or an $\hat{S}$**-translation**, of R_i. If $\hat{S} = \{s\}$, we simply write s**-translation**. If $d > 0$ we call the translation **positive**. We consider this the positive direction because it represents an increased preference for s relative to other sites. In fact, for any $x \in \mathbb{R} \times S$ and any $s \in S$, the lower contour set of $R_i^{s,d}$ at (x_i, s_i) contains the lower contour set of R_i at (x_i, s_i).

Note that $R_i \in \mathcal{D}$ does not guarantee $R_i^{s,d} \in \mathcal{D}$. In particular, when $\mathcal{D}$ satisfies *zero commodity indifference*, the only translations that remain in the domain are the identity translations. This is not a problem; if a rule is *strategy-proof* on $\mathcal{D}^N \cup \left\{\left(R_i^{s,d}, R_{-i}\right)\right\}$ then it is *strategy-proof* on $\mathcal{D}^N$. Moreover, the translations we introduce in the course of proving the results do not upset the existence of equilibria.

Now we collect in a theorem the salient properties of p^*.

Theorem 3. *Let $R \in \mathcal{R}^N$ be a profile on which F^* is defined. Fix $i \in N$ and $s \in S$. Define the function π for each $a \in \mathbb{R}$ by $\pi(a) := p^*(R_i^{s,a}, R_{-i})$.*

Property 1: *π_s is non-decreasing*

Property 2: *If $s \notin D_S(R_i, p^*(R))$, then there exists $\bar{d} > 0$ such that for each $d < \bar{d}$, $\pi(d) = \pi(0)$.*

3.3.2. *Incentive Properties of Min-price Rules.* We study a standard incentive compatibility property: no group of agents should strictly benefit by reporting false preferences.

Weak Group-strategy-proofness (w-GStP): For each $R \in \mathcal{R}^N$, each group $N' \subseteq N$, and each partial profile of preferences $\hat{R}_{N'} := \left(\hat{R}_i\right)_{i\in N'} \in \mathcal{R}^{N'}$, there is an agent $k \in N'$ such that

$$\varphi_k(R)\ R_k\ \varphi_k\left(\hat{R}_{N'}, R_{N\setminus N'}\right).$$

The tools of Theorem 3 make intuitive why min-price rules should have such nice incentive properties. Agents have limited influence over the prices of each site, and what influence they do have is the "appropriate" kind. Most crucially, we show that if an agent causes the price of a site to decrease, then the agent must abandon consumption at this site. This is due to the indivisibilities in the problem and is a major difference between equilibrium here and equilibrium in classical exchange economies.

To prove that F^* satisfies *w-GStP*, the following lemma is useful.

Lemma 2. *Let $N' \subseteq N$ and let $d := 2\max_{s\in S}\{e_s\}$. Let $(s_j)_{j\in N'}$ be an arbitrary list of sites. There is a site $t \in (s_j)_{j\in N'}$ such that*

$$p_t^*\left(\left(R_j^{s_j,d}\right)_{j\in N'}, R_{N\setminus N'}\right) \geq p_t^*(R).$$

Theorem 4. *F^* is weakly group-strategy-proof.*

Proof. In what follows, the preferences of agents $N \setminus N'$ are held constant and therefore we suppress their notation.

Let $\mathcal{D}$ be a domain where F^* is defined, let $f \in F^*$, and let $R \in \mathcal{D}$. To arrive at a contradiction, assume there is a set $N' \subseteq N$ and a partial profile $\hat{R} := \left(\hat{R}_j\right)_{j\in N'}$ such that for each $k \in N'$,

$$(x_k, s_k) := f_k\left(\hat{R}\right) \quad P_k \quad f_k(R).$$

Note that $f\left(\hat{R}\right)$ is an equilibrium for $R^d := \left(R_j^{s_j,d}\right)_{j\in N'}$. Therefore $p^*\left(R^d\right) \leq p^*\left(\hat{R}\right)$. Since d is large, each $k \in N'$, given preferences R_k^d, will choose only s_k at prices $p^*\left(R^d\right)$. Thus, for each $k \in N'$, there is $\bar{x}_k$ satisfying $\bar{x}_k \geq x_k$ such that

$$f_k\left(R^d\right) = \left(\bar{x}_k, s_k\right).$$

Therefore, R^d is also a joint manipulation for group N'. Now we apply Lemma 2: there is an site $t \in (s_j)_{j\in N'}$ such that

$$p_t^*\left(R^d\right) \geq p_t^*\left(R\right),$$

a contradiction. ■

4. Characterization

In this section we provide a characterization of min-price rules in terms of appealing properties. We believe that these rules have many interesting properties beyond what is written here; however, our primary focus is on preference elicitation in the presence of a weakened form of *anonymity*. Fix a single-valued rule φ whose properties we enumerate below. We show that these properties imply φ is generically a selection from a min-price rule.

The first property is implied by, and is much weaker than *weak group-strategy-proofness*.

Strategy-proofness (StP): For each $R \in \mathcal{R}^N$, each $i \in N$, each preference relation $\hat{R}_i \in \mathcal{R}$,

$$\varphi_i\left(R\right) R_i\ \varphi_i\left(\hat{R}_i, R_{-i}\right).$$

In this model, as in many others, we may adapt the usual sequential priority procedure to both elicit preferences truthfully and achieve *Pareto efficiency*. The inequity of such rules is extreme and therefore they are inappropriate for the applications we have envisioned. We insist upon a form of *anonymity* that requires an agent's *welfare* depend only on his preferences and the unordered list of preference relations present in the economy. Note that this is weaker than the usual form, which requires an agent's *bundle* depend only on his preferences and the unordered list of preference relations present in the economy.

Welfare Anonymity (W-Anon): Let $R \in \mathcal{R}^N$ and let $\sigma : N \to N$ be a bijection. Let $i \in N$ and $\sigma(j) = i$. Then

$$\varphi_i\left(R\right) I_i\ \varphi_j\left((R_{\sigma(k)})_{k\in N}\right).$$

We consider the following property a regularity condition. Consider a convergent sequence of profiles.[6] Suppose the rule chooses the same allocation for all of the profiles on the sequence. Then in the limit profile, the agents are indifferent between what the rule chooses at the limit and what it has chosen all along the sequence.

[6]Since the consumption space is compact and preferences continuous, the Hausdorff distance ρ on the graphs of any pair R and R' of preference relations generates a metric topology on the space of continuous preference relations.

Constant Sequence Continuity (w-Cont): Let $(R^n)^{n\in\mathbb{N}} \subset \mathcal{R}^N$ be a sequence converging to R. Assume there is an allocation (x,α) such that for each $n \in \mathbb{N}$,

$$\varphi\left(R^n\right) = (x,\alpha)\,.$$

Then for each agent i, $(x_i, \alpha(i))\ I_i\ \varphi_i\,(R)$.

In a model as rich as this, we should not expect *StP*, *w-Cont*, and *W-Anon* to identify a single rule. Rather than introduce further properties, however, we study the consequences of a rule being *maximal* in the properties already given. This is a principle of second-best efficiency and takes the place of imposing Pareto Efficiency. In general, we may define

Strong Undomination in $\mathfrak{C}$: Fix a class of rules $\mathfrak{C}$. Say that φ is *strongly undominated in* $\mathfrak{C}$ if for each $\psi \in \mathfrak{C}$ and each $R \in \mathcal{R}^N$

$$\psi_i\,(R)\ P_i\ \varphi_i\,(R) \implies \exists j,\ \varphi_j\,(R)\ P_j\ \psi_j\,(R)\,.$$

We require that φ be *strongly undominated* in the class of rules satisfying our previous properties.

Strong Undomination in W-Anon, StP, w-Cont: Rule φ is *strongly undominated* in the class of rules satisfying *welfare anonymity*, *strategy-proofness*, and *constant sequence continuity*. Henceforth we refer to this property simply as *strong undomination*.

We may now state the characterization, the proof of which is in the appendix.

Theorem 5. *Let φ be a single-valued rule. Assume φ is strategy-proof, welfare anonymous, constant sequence continuous, and strongly undominated in these properties. Then there is an open and dense set $\mathcal{D}^* \subset \mathcal{R}^N$ such that for each $R \in \mathcal{D}^*$, $\varphi(R) \in F^*(R)$.*

5. Implementation

An auction is a type of game form used to implement allocation rules that are based on prices. The messages of an auction are demand schedules. In this model, a demand schedule for agent i is a function $D_i : \mathbb{R}^S_+ \to 2^S$ such that there exists $R_i \in \mathcal{R}$ satisfying $D_i\,(\cdot) = D_S\,(R_i, \cdot)$. The set of demand schedules is $\mathfrak{D}$. It is desirable that, rather than reporting their entire demand schedule, agents instead report their demands in response to a smaller list of prices. Typically, a price $q \in \mathbb{R}^S$ is announced and the reported demands, $(D_i(q))_{i\in N}$, then determine the next price asked, thus making the auction game an extensive form. We shall assume that the auction proceeds in continuous time, therefore the price dynamic is described by a differential equation. Let $\delta > 0$. We propose a price dynamic whose time derivative, $\pi : \mathfrak{D}^N \times \mathbb{R}^S \to \mathbb{R}^S$, satisfies for each $D \in \mathfrak{D}^N$, each $q \in \mathbb{R}^S$, and each $s \in S$, $\pi(D, q) \in \{0, \delta\}$. Therefore, to define $\dot{p}$ it remains to determine the rule for choosing which set of sites will have their prices rising.

Fix $D \in \mathfrak{D}^N$. For each site $s \in S$, each $N' \subseteq N$, and each $q \in \mathbb{R}^S$, let

$$\rho_s(N', q) := \min\{|\{i \in N' : \ s \in D_i(q)\}|, c_s(q)\}.$$

A set of sites $\hat{S} \subseteq S$ is **overdemanded at** q if the set $\hat{N} := \left\{i \in N : \ D_i(q) \subseteq \hat{S}\right\}$ satisfies $\left|\hat{N}\right| > \sum_{s \in \hat{S}} \rho_s\left(\hat{N}, q\right)$. Let $N' \subseteq N$ be arbitrary. Let $S' := \cup_{i \in N'} D_i(q)$. Then if there are no overdemanded sites at q,

$$|N'| \leq \sum_{s \in S'} \rho_s(N', q) = \sum_{s \in S} \rho_s(N', q),$$

where the equality follows from the fact that, for each $r \in S \setminus S'$, $\rho_r(N', q) = 0$. Thus, the Matroid Partition Theorem implies that q is an equilibrium price satisfying demands $(D_i(q))_{i \in N}$.

The auction will proceed by raising the price of *some*, but not all overdemanded sets. In particular, a set $\hat{S}$ is **minimally overdemanded** if it is overdemanded and if, for each $\tilde{S} \subsetneq \hat{S}$, $\tilde{S}$ is not overdemanded. A site $s \in S$ is **minimally overdemanded** if it is a member of a minimally overdemanded set. Thus we set

$$\pi_s(D, q) := \begin{cases} \delta & s \text{ is minimally overdemanded} \\ 0 & \text{otherwise.} \end{cases}$$

Let $R \in \mathcal{R}^N$ and, for each $i \in N$, let $D_i(\cdot) := D_S(R_i, \cdot)$. Let $p : \mathbb{R} \to \mathbb{R}^S$ be given by the differential equation $\dot{p}(t) = \pi(D, p(t))$ with initial condition $p(0) = w^{-1}e = (w^{-1}e_1, \ldots, w^{-1}e_{|S|})$. We now show that the price dynamic thus defined converges in finite time to the minimal Walrasian price.

Proposition 2. *There exists $T \in \mathbb{R}_+$ such that $p(T) = p^*(R)$.*

Proof. We first prove the following claim:

Claim 1. For each $t \in \mathbb{R}$, $p(t) \leq p^*(R)$.

Proof. Let $p \leq p^*(R)$ and let $S^{eq} := \{s \in S : p_s = p_s^*(R)\}$. Let $N^{eq} := \{i \in N : D_i(p) \cap S^{eq} \neq \emptyset\}$. If $s \in S^{eq}$ and $s \in D_i(p)$ then $s \in D_i(p^*(R))$. Therefore, for each $N' \subseteq N^{eq}$ and each $s \in S^{eq}$, $\rho_s(N', p) = \rho_s(N', p^*(R))$. For each $r \notin S^{eq}$, $\rho_r(N', p^*(R)) = 0$. Let $S' \subseteq S^{eq}$ and $N' := \{i \in N : \ D_i(p) \subseteq S'\}$. Since $N' \subseteq N^{eq}$, if it were the case that S' is overdemanded at p, then

$$|N'| > \sum_{s \in S'} \rho_s(N', p) = \sum_{s \in S} \rho(N', p^*(R)),$$

contradicting, via the Matroid Partition Theorem, the fact that $p^*(R)$ is an equilibrium price. Thus, neither S^{eq} nor any of its subsets are underdemanded at p.

Now let $N' \subseteq N^{eq}$ be arbitrary. Since $p^*(R)$ is an equilibrium price we deduce

$$|N'| \leq \sum_{s \in S} \rho_s(N', p^*(R)) = \sum_{s \in S^{eq}} \rho_s(N', p^*(R)) = \sum_{s \in S^{eq}} \rho_s(N', p), \tag{4}$$

where the initial inequality is from the Matroid Partition Theorem. Now let $\hat{S} \supseteq S^{eq}$ and $\hat{N} := \left\{i \in N : \ D_i(p) \subseteq \hat{S}\right\}$. If $s \in S^{eq}$, then $\rho_s\left(\hat{N}, p\right) = \rho_s\left(\hat{N} \cap N^{eq}, p\right)$. Therefore, if $\hat{S}$ is overdemanded, line 4 and the Matroid Partition Theorem yield

$$\begin{aligned}
\left|\hat{N}\right| &> \sum_{s \in S \setminus S^{eq}} \rho_s\left(\hat{N}, p\right) + \sum_{s \in S^{eq}} \rho_s\left(\hat{N}, p\right) \\
&= \sum_{s \in S \setminus S^{eq}} \rho_s\left(\hat{N}, p\right) + \sum_{s \in S^{eq}} \rho_s\left(\hat{N} \cap N^{eq}, p\right) \\
&\geq \sum_{s \in S \setminus S^{eq}} \rho_s\left(\hat{N}, p\right) + \left|\hat{N} \cap N^{eq}\right|.
\end{aligned}$$

Therefore, $\left|\hat{N} \setminus N^{eq}\right| > \sum_{s \in S \setminus S^{eq}} \rho_s\left(\hat{N}, p\right) \geq \sum_{s \in S \setminus S^{eq}} \rho_s\left(\hat{N} \setminus N^{eq}, p\right)$, where the last inequality is by monotonicity of $\rho(\cdot, p)$. We conclude that $S \setminus S^{eq}$ is overdemanded and, furthermore, that for each $s \in S^{eq}$, s is not minimally overdemanded.

Since the price path $p(\cdot)$ is continuous and since $p(0) = w^{-1}e$, if there exist $t \in \mathbb{R}$ and $s \in S$ such that $p_s(t) > p_s^*(R)$, then there are $t' < t$ and $s' \in S$ such that $p(t') \leq p^*(R)$ and $p_{s'}(t') = p_{s'}^*(R)$. What we deduced in the preceding paragraphs then implies that for $t'' \geq t'$, $\dot{p}_{s'}(D, p(t'')) = 0$, a contradiction. □

Note that if $t' > t$ and $p(t') = p(t)$, then $p(t) \in \mathbb{P}(R)$. Contrapositively, if $p(t) \notin \mathbb{P}(R)$, then for each $t' > t$, $p(t') \gneq p(t)$, and the rate of increase is bounded away from zero.

Therefore, since $p(\cdot)$ is bounded above by $p^*(R)$, there is some finite $T \in \mathbb{R}$ such that for each $t' > T$, $p(t') = p(T)$. It follows that $p(T) \in \mathbb{P}(R)$, so $p(T) \geq p^*(R) \geq p(T)$ and the result follows. ■

The following theorem is now an immediate consequence of the Revelation Principle:

Theorem 6. *The auction using price dynamic $p(\cdot)$ implements F^* in dominant strategies.*

6. Conclusion

We have introduced a new model with a wide range of applications. Our solution has also provided novel insights for existing models with potential applications to housing problems (when many agents will share a house), course assignment with variable tuition (many agents take a course), and possibly many-to-one matching with general preferences and a continuum of transfers.

From a theoretical perspective, it seems vital to understand why the extremal element of a lattice should so often generate a *strategy-proof* rule. Leonard (1983) gave excellent insight into this, via the dual of a linear program, for the quasilinear case. Similar reasoning should hold here, but our comparative statics provide intuition of a different nature. The minimality of p^* ensures that it will respond in "the right way" to changes in preferences. Agents would surely like the price to decrease further, but the result would not be an equilibrium and therefore could not be the result of the rule unless someone's preferences changed. But the required preference change is precisely that which would make the manipulating agent cease to consume the cheaper resource.

Appendix A. Proofs For Section 3

When seeking equilibria, we may restrict our search to triples (p, x, α) such that each agent i consumes a bundle $(x_i, \alpha(i))$ with the property that $p_{\alpha(i)} x_i = w$. Such allocations can be identified by a price vector and site assignment alone. We therefore define a function $|\cdot, \cdot| : \mathbb{R}^S \times S^N \to \mathbb{R}^N \times S^N$ that maps each pair of a price vector and site assignment to an allocation as follows. Let $(p, \alpha) \in \mathbb{R}^S \times S^N$. Let $x \in \mathbb{R}^N$ satisfy, for each $i \in N$,

$$x_i = \frac{w}{p_{\alpha(i)}}.$$

Then $|p, \alpha| := (x, \alpha)$. For each $i \in N$, let $|p, \alpha(i)| := (x_i, \alpha(i))$. Agent i's bundle $|p, \alpha(i)|$ need not be an optimal choice for him from the p budget set.

A.1. **Some topological properties.** For simplicity, fix the consumption space

$$Z := \{(x_0, s_0) : 0 \le x_0 \le e_{s_0}\}.$$

Endow Z with the metric ρ defined for each pair $((x_0, s_0), (y_0, t_0))$ by

$$\rho\left[(x_0, s_0), (y_0, t_0)\right] = \begin{cases} |x_0 - y_0| & s_0 = t_0 \\ 2\max_{s \in S} e_s & s_0 \neq t_0. \end{cases}$$

Let the metric on $Z \times Z$ be given by the maximum of ρ calculated component-wise. A preference relation is a closed subset $R \subset Z \times Z$. Since $Z \times Z$ is compact, the Hausdorff distance δ, calculated with respect to the product metric is a metric for the space of preference relations $\mathcal{R}$. In fact, the topology induced by δ is precisely the topology of closed-convergence. See Hildenbrand (1974). Endow $\mathcal{R}^N$ with the product topology.

Lemma 3. *Let $p^n \to p \in \mathbb{R}^S_{++}$ and $R^n_i \to R_i$. There exists $\bar{n} \in \mathbb{N}$ such that for each $n \ge \bar{n}$, $D_S(R^n_i, p^n) \subseteq D_S(R_i, p)$.*

Proof. Let $s_0 \notin D_S(R_i, p)$. Let

$$\varepsilon := \max\left\{\frac{w}{p_t} - x_0 : (x_0, t)\ I_i\ \left(\frac{w}{p_{s_0}}, s_0\right)\right\}.$$

Since $s_0 \notin D(R_i, p)$, $\varepsilon > 0$. Assume $(z_0, r_0)\ I_i\ \left(wp_{s_0}^{-1}, s_0\right)$ satisfies $wp_{r_0}^{-1} - z_0 = \varepsilon$. For each $n \in \mathbb{N}$, let $(z(n), r_0)\ I_i\ \left(w\left[p^n_{s_0}\right]^{-1}, s_0\right)$. Since $p^n \to p$ and p has no zero components, $w\left[p^n_{r_0}\right]^{-1} \to wp_{r_0}^{-1}$. Therefore, since preferences are continuous, there is $n^1 \in \mathbb{N}$ such that for each $n > n^1$, $w\left[p^n_{r_0}\right]^{-1} - z(n) > \varepsilon/2$. For each n, let $x(n)$ satisfy $(x(n), r_0)\ I^n_i\ \left(w\left[p^n_{s_0}\right]^{-1}, s_0\right)$. There is an $n^2 \in \mathbb{N}$ such that for each $n > n^2$, $|z(n) - x(n)| < \varepsilon/2$. Let $\bar{n}_{s_0} = \max\{n^1, n^2\}$. Then for each $n > \bar{n}_{s_0}$.

$$\begin{aligned} \frac{w}{p^n_{r_0}} - x(n) &\ge \frac{w}{p^n_{r_0}} - z(n) + z(n) - x(n) \\ &> \frac{\varepsilon}{2} + z(n) - x(n) > 0. \end{aligned}$$

Therefore, for each $n \ge \bar{n}_{s_0}$, $s_0 \notin D_S(R^n_i, p^n)$.

Let $\bar{n} := \max\{\bar{n}_{s_0} : s_0 \notin D_S(R_i, p)\}$. We have shown that for each $n > \bar{n}$, $S \setminus D_S(R_i, p) \subseteq S \setminus D_S(R_i^n, p^n)$ and the result follows. ■

For each $K \in \mathbb{N}$, let $\mathbf{1} \in \mathbb{R}^K$ be the vector $(1, 1, \ldots, 1)$. We define the *limit inferior* of a sequence $x^n \in \mathbb{R}^K$ component-wise: for each k, let $\underline{x}_k := \liminf x_k^n$. Then define $\liminf x^n := \underline{x}$. Note that

$$\liminf x^n = \lim_{n\to\infty}\left[\inf_{\leq}\left\{x^{\tilde{n}} : \tilde{n} \geq n\right\}\right],$$

since the interior infimum can be found component-wise and still results in a non-decreasing sequence in the vector order. The *limit superior* is symmetric.

Corollary 1. *p^* is lower semi-continuous.*

Proof. Let $R^n \to R$. Let $p := \liminf p^*(R^n)$. Let $s \in S$. There is a sub-sequence $R^{\sigma(n)}$ such that $\lim p_s^*\left(R^{\sigma(n)}\right) = p_s$. Let $p^1 := \liminf p^*\left(R^{\sigma(n)}\right)$. For $t \neq s$, there is a further sub-sequence $R^{\tau(n)}$ such that $\lim p_t^*\left(R^{\tau(n)}\right) = p_t^1$. By repeating the process, we find a sequence $R^{\nu(n)}$ and a price list p^s such that $\lim p^*\left(R^{\nu(n)}\right) = p^s$ and $p_s^s = p_s$. Note that the equilibrium price of each site t is bounded below by $we_t^{-1} > 0$. Therefore, $p^s \in \mathbb{R}_{++}^S$. Since S^N is finite, there is a site assignment $\alpha \in S^N$ and a further sub-sequence $R^{\tilde{\nu}(n)}$ such that for each n, $\alpha \in \mathcal{A}^*\left(R^{\tilde{\nu}(n)}\right)$. By Lemma 3, for each $i \in N$, $\alpha(i) \in D_S(R_i, p^s)$. Therefore, $|p^s, \alpha|$ is an equilibrium for R and $p^s \in \mathbb{P}(R)$. Since s was arbitrary, the lower semi-lattice property of $\mathbb{P}(R)$ (Theorem 1) implies that $p \in \mathbb{P}(R)$. Finally, minimality of p^* yields

$$p^*(R) \leq p = \liminf p^*(R^n).$$

■

A.2. **Blocking via Indifference.** For convenience, we restate the Lemma.

Lemma 1. *Let $R \in \mathcal{R}^N$ and $(x, \alpha) \in F^*(R)$ be supported by prices p. Assume that s is not exhausted at equilibrium (p, x, α). Thenk either $s \notin \cup_{i\in N} D_S(R_i, p)$ or s is blocked via indifference by a site t that is exhausted at (p, x, α).*

Proof. Let $|p, \alpha| \in F^*(R)$. For each $S' \subseteq S$ and each $\varepsilon > 0$ define

$$p(S', \varepsilon) := \begin{cases} p_s & s \notin S' \\ p_s - \varepsilon & s \in S'. \end{cases}$$

Given $\varepsilon > 0$, we find a set $\hat{S} \subseteq S$ such that perturbed prices $p(V, \varepsilon)$ induce preference chains that go from an exhausted site to a non-exhausted site. We collect these chains in set $\mathbb{C}(\varepsilon)$. The chains leading to a particular unexhausted site $s \in S$ are denoted $\mathbb{C}(\varepsilon, s)$. We fix one particular s and find that for each $\varepsilon > 0$, $\mathbb{C}(\varepsilon, s)$ is non-empty. A limiting argument then completes the proof.

Let s be a site that is not exhausted at $|p, \alpha|$. Suppose that for each $\varepsilon > 0$, there is no agent $j \notin \alpha^{-1}(s)$ such that

$$|p(\{s\}, \varepsilon), \alpha(i)| \; P_j \; |p(\{s\}, \varepsilon), \alpha(j)| = |p, \alpha(j)| \,. \tag{5}$$

If $\alpha^{-1}(s) = \emptyset$, then clearly $s \notin \cup_{i \in N} D_S(R_i, p)$. If there is $i \in \alpha^{-1}(s)$, then for ε sufficiently small, s is not exhausted at $|p(\{s\}, \varepsilon), \alpha|$. We have found another equilibrium with a lower price, a contradiction. Assume then that $\alpha(i) = s$ and therefore there exists $j \notin \alpha^{-1}(s)$ satisfying (5). If $t := \alpha(j)$ is exhausted at $|p, \alpha|$, then set $(i, j) \in \mathbb{C}(\varepsilon, s)$. If t is not exhausted at $|p, \alpha|$, then we look for an agent $k \notin \alpha^{-1}(\{s, t\})$ and an agent $k' \in \{i, j\}$ such that

$$|p(\{s, t\}, \varepsilon), \alpha(k')| \; P_k \; |p(\{s, t\}), \alpha(k)| = |p, \alpha(k)| \,.$$

If we can find such an agent, there are two cases: if $k' = i$ then write $\langle (k, i), (j, i) \rangle$, otherwise write $\langle (k, j, i) \rangle$. That is, append k to the chain that he extends. In case $k' = i$, k creates a new path to i. Retain all the chains previously constructed by collecting them in brackets, $\langle \cdots \rangle$.

Claim. Suppose we have iterated the above procedure and arrived at chains $\langle (j, \ldots, i), \ldots, (k, \ldots, i) \rangle$ (by construction, all chains lead to i). Let

$$V := \{s \in S : \; \exists l \in (l', \ldots, i) \in \langle (j, \ldots, i), \ldots, (k, \ldots, i) \rangle, \; \alpha(l) = s\} \,.$$

Assume that none of the sites in V are exhausted at $|p, \alpha|$. Then there is an agent $l \notin \alpha^{-1}(V)$ and an agent $l' \in \alpha^{-1}(V)$ such that

$$|p(V, \varepsilon), \alpha(l')| \; P_l \; |p(V, \varepsilon), \alpha(l)| = |p, \alpha(l)| \,.$$

Proof. Denote the economy by $\mathcal{E}$. Let $\hat{\mathcal{E}}$ denote the reduced economy with only sites V and agents $\hat{N} := \alpha^{-1}(V)$. We map this economy into the model of one-to-one matching with transfers and and denote this image $\hat{\mathcal{E}}^{\star}$.

Let $\underline{c} \in \mathbb{N}^V$ satisfy, for each $\hat{s} \in V$, $\underline{c}_{\hat{s}} = |\alpha^{-1}(\hat{s})|$. Construct the set $V^\star$ from V by having each site $\hat{s}$ exist as $\underline{c}_{\hat{s}}$ identical copies. Copies of s are denoted s^a, s^b, etc. Agents i's preferences $R^{\star i}$ over $\mathbb{R} \times V^\star$ are defined in the natural way: for each $x \in \mathbb{R}$, $\left\{\hat{s}^a, \hat{s}^b\right\} \subseteq \{\hat{s}\}^\star$ implies $(x, \hat{s}^a)\ \hat{I}_i\ \left(x, \hat{s}^b\right)$ and otherwise $\hat{R}_i$ respects R_i. We define an endowment vector $e^\star$ for each $\hat{s}^a \in \{\hat{s}\}^\star$ as

$$e^\star_{\hat{s}^a} = \frac{w}{p_{\hat{s}}(V, \varepsilon)}.$$

Finally define capacity vector $c^\star \in \mathbb{N}^{V^\star}$ by $c^\star = (1, 1, \ldots, 1)$. Let $\hat{\mathcal{E}}^\star$ be the economy consisting of sites $V^\star$ with capacities $c^\star$, endowments $e^\star$, and agents $\hat{N}$ having preferences $R^{\star\hat{N}}$. This economy admits an equilibrium $|p^\star, \alpha^\star|$ by adapting $|p, \alpha|$ in the obvious way. Then by Lemma 3 in Demange and Gale (1985), it admits an equilibrium $|q^\star, \beta^\star|$ such that for at least one site r_l,

$$\frac{w}{q^\star_{r_l}} = e^\star_{r_l},$$

implying that $q^\star_{r_l} = p_r(V, \varepsilon)$. Moreover the lattice structure allows us to assume $q^\star \leq p^\star$.

Let $\left\{t^a, t^b\right\} \subseteq \{t\}^\star$. By the construction of $R^\star$, and since the site-assignment is now one-to-one, $q^\star_{t^a} = q^\star_{t^b}$. Thus, $q_t := q^\star_{t^a}$ is well-defined. Define β for each site $\hat{s} \in V$ by $\beta^{-1}(\hat{s}) := \beta^{\star-1}(\{\hat{s}\}^\star)$. We have constructed an equilibrium $|q, \beta|$ for $\hat{\mathcal{E}}$ such that for each $\hat{s} \in V$, $q_{\hat{s}} \leq p_{\hat{s}}$. Moreover,

$$q_r = p_r(V, \varepsilon) < p_r,$$

Extend q for each $t \in S \setminus V$ by setting $q_t := \infty$. Let

$$\gamma(j) := \begin{cases} \alpha(j) & j \in N \setminus \hat{N} \\ \beta(j) & j \in \hat{N}. \end{cases}$$

By construction $|p \wedge q, \gamma|$ is a feasible allocation for $\mathcal{E}$. If there exists no agent l as in the claim, then $|p \wedge q, \gamma|$ is an equilibrium for $\mathcal{E}$, contradicting the minimality of p for R. Thus the claim is shown. □

The number of chains is finite, and the length of each chain is finite. Thus the procedure will terminate, yielding a family of chains $\langle (j^1, \ldots, i), (j^2, \ldots, i), \ldots, (j^m, \ldots, i) \rangle$. Assume there is no chain $\left(j^k, \ldots, i\right)$ such that $\alpha(j^k)$ is exhausted at $|p, \alpha|$. Then, by construction, for each $\hat{s} \in S$, there exists $l \in (j^1, \ldots, i)$ such that $\alpha(l) = \hat{s}$. The same holds for each $j^k \in \{j^1, j^2, \ldots, j^m\}$. Each chain covers all sites. But we may apply the proof of the claim to find an equilibrium price $q \leq p$ such that $q \neq p$. This contradicts the minimality of p. We conclude that there is a chain $\left(j^k, \ldots, i\right)$ such that $\alpha(j^k)$ is exhausted at $|p, \alpha|$. Therefore, $\left(j^k, \ldots, i\right) \in \mathbb{C}\left(\varepsilon, s\right)$.

We have shown that if $\alpha(i)$ is not exhausted at $|p, \alpha|$ then for each $\varepsilon > 0$, the family $\mathbb{C}\left(\varepsilon, \alpha(i)\right)$ associated with $|p, \alpha|$ is non-empty. Let ε^n be a sequence decreasing to zero. Recall we have constructed the chains in $\mathbb{C}\left(\varepsilon^n, s\right)$ such that they do not repeat sites. There are a finite number of such chains emanating from an exhausted site and ending at $\alpha(i)$. Thus, there exists one such chain $(j, \ldots, i)$ and a sub-sequence $\varepsilon^{\sigma(n)}$ such that for each n, $(j, \ldots, i) \in \mathbb{C}\left(\varepsilon^{\sigma(n)}, s\right)$. By continuity of preferences, $(j, \ldots, i) \in \mathbb{C}\left(i, 0\right)$, and the lemma is proved. ■

Proposition 1. *For each profile R, there is a balanced site matching $\alpha \in \mathcal{A}^*(R)$.*

Proof. Without loss of generality, we may normalize the endowment vector to $e = \mathbf{1}$ and the distributed numeraire to $w = 1$. In this case, for each $s \in S$, $c_s\left[p^*\left(R\right)\right] = \lfloor p_s^*\left(R\right) \rfloor$. Let $R \in \mathcal{R}^N$ and $p := p^*(R)$. It suffices to find a site assignment $\alpha \in \mathcal{A}^*\left(R\right)$ such that, for each $s \in S$,

$$\left|\alpha^{-1}\left(s\right)\right| + 1 \geq p_s. \tag{6}$$

For the remainder of this paragraph, we prove that (6) is sufficient. There are two cases. In the first, p_s is an integer. Then $p_s = c_s\left(p\right)$. Feasibility implies that $c_s(p) \geq |\alpha^{-1}\left(s\right)|$. Together with (6), we have

$$c_s\left(p\right) \geq \left|\alpha^{-1}\left(s\right)\right| \geq c_s\left(p\right) - 1,$$

and so $|\alpha^{-1}(s)| \in \{c_s(p), c_s(p) - 1\}$, as desired. If p_s is not an integer, then (6) is a strict inequality. Thus,

$$c_s\left(p\right) \geq \left|\alpha^{-1}\left(s\right)\right| > \lfloor p_s \rfloor - 1 = c_s\left(p\right) - 1,$$

which implies, since $|\alpha^{-1}(s)|$ is an integer, that $c_s\left(p\right) = |\alpha^{-1}(s)|$.

Let $p := p^*(R)$ and $\alpha \in \mathcal{A}^*(R)$. Suppose there is a site s violating inequality (6). Then s is not exhausted at $|p, \alpha|$. Moreover, if $\alpha^{-1}(s) = \emptyset$, then by definition of equilibrium, $p_s = 1$, and inequality (6) then implies $|\alpha^{-1}\left(s\right)| < 0$, which is impossible. Therefore s is blocked via indifference by an exhausted site s^0. We have

$$\left|p, \alpha(i^k)\right| \; I_{i^k} \cdots I_{i^3} \; \left|p, \alpha(i^2)\right| \; I_{i^2} \; \left|p, \alpha(i^1)\right|,$$

with $\alpha(i^k) = s^0$ and $\alpha(i^1) = s$. Construct site-assignment β by moving agents along this indifference path: for $l \in \{2, \dots, k\}$, $\beta(i^l) = \alpha(i^{l-1})$, and for all other agents, $\beta(i) = \alpha(i)$. Clearly, $\beta \in \mathcal{A}^*(R)$. Since site s^0 was exhausted under α,

$$\left|\alpha^{-1}(s^0)\right| \frac{w}{p_{s^0}} = e_{s^0},$$

which, under our normalization, yields $|\alpha^{-1}(s^0)| = p_{s^0}$. Thus, $|\beta^{-1}(s^0)| + 1 = p_{s^0}$. For each $t \in S \setminus \{s, s^0\}$, $|\beta^{-1}(t)| = |\alpha^{-1}(t)|$. Finally, $|\beta^{-1}(s)| = |\alpha^{-1}(s)| + 1$.

If site s still violates inequality (6) under matching β, we repeat the exercise. In the next iteration, s is blocked via indifference by site $s^1 \neq s^0$ with s^1 exhausted at $|p, \beta|$. Proceeding thus, we generate a list of sites $\left\{s^0, s^1, \dots, s^k\right\}$ and a site assignment $\gamma \in \mathcal{A}^*(R)$ such that for each $s^l \in \left\{s^0, s^1, \dots, s^k\right\}, \left|\gamma^{-1}\left(s^l\right)\right| + 1 = p_{s^l}$ and, moreover, $|\gamma^{-1}(s)| = |\alpha^{-1}(s)| + k + 1$. Finally, for all other sites t, $|\gamma^{-1}(t)| = |\alpha^{-1}(t)|$. Eventually, either set s is no longer blocked via indifference by an exhausted site, or it no longer violates the inequality. The former case is ruled-out by Lemma 1. Thus the latter case is true and the corollary is proved. ■

A.3. **Proof of Theorem 3.** For convenience, we restate the theorem:

Theorem 3. *Let $R \in \mathcal{R}^N$ be a profile on which F^* is defined. Fix $i \in N$ and $s \in S$. Define the function π for each $a \in \mathbb{R}$ by $\pi(a) := p^*(R_i^{s,a}, R_{-i})$.*

Property 1: *π_s is non-decreasing*
Property 2: *If $s \notin D_S(R_i, p^*(R))$, then there exists $\bar{d} > 0$ such that for each $d < \bar{d}$, $\pi(d) = \pi(0)$.*

We prove Theorem 3 in parts. We can simplify notation as follows: for each $r \in \mathbb{R}$, $R^r := (R_i^{s,r}, R_{-i})$.

Proof of Property 2. Since $s \notin D_S(R_i, \pi(0))$, there is an open set $U \subset \mathbb{R}$ containing zero such that for each $\varepsilon \in U$, $s \notin D_S(R_i^{s,\varepsilon}, \pi(0))$. For each $\alpha \in \mathcal{A}^*(R)$, $|\pi(0), \alpha|$ is an equilibrium for R^ε. Thus $\pi(\varepsilon) \leq \pi(0)$.

Case 1: There is an open neighborhood V containing 0 such that for each $\varepsilon \in V$, $s \notin D_S(R_i^{s,\varepsilon}, \pi(\varepsilon))$. Then if $|\pi(\varepsilon), \beta|$ is an equilibrium for R^ε, $\beta(i) \neq s$. Since R_i and $R_i^{s,\varepsilon}$ agree on $\mathbb{R} \times S \setminus s$, and since $R_{-i}^\varepsilon = R_{-i}$, $|\pi(\varepsilon), \beta|$ is also an equilibrium for R . Thus by minimality, $\pi(0) \leq \pi(\varepsilon)$ and we conclude that $\pi(\varepsilon) = \pi(0)$.

Case 2: There is a sequence ε^n converging to zero such that for each $n \in \mathbb{N}$, $s \in D_S\left(R_i^{s,\varepsilon^n}, \pi(\varepsilon^n)\right)$. We showed that for each $\varepsilon > 0$ sufficiently small, $\pi(0) \geq \pi(\varepsilon)$. Combined with Corollary 1, we may write

$$\pi(0) \geq \limsup_{n\to\infty} \pi(\varepsilon^n) \geq \liminf_{n\to\infty} \pi(\varepsilon^n) \geq \pi(0).$$

Therefore $\lim_{n\to\infty} \pi(\varepsilon^n) = \pi(0)$. By Lemma 3, $s \in D_S(R_i, \pi(0))$, a contradiction.

We have shown that if $s \notin D_S(R_i, p^*(R))$, then for d sufficiently small, $\pi(d) = \pi(0)$. For each such d, $\pi(0) \in \mathbb{P}\left(R_i^{s,d}, R_{-i}\right)$, and therefore $\pi(0) \geq \pi(d)$. Define $\bar{d}$ as the unique number satisfying $D_S\left(R_i^{s,\bar{d}_i}, p^*(R)\right) = D_S(R_i, p^*(R)) \cup \{s\}$. Let d^n be an increasing sequence converging to $\bar{d}$. Then

$$\pi(0) \geq \limsup \pi(d^n) \geq \liminf \pi(d^n) \geq \pi(0),$$

where the final inequality is from the lower semi-continuity of p^*. ■

Lemma 4. *Assume $\alpha(i) = s$. Then for each $d > 0$, each $\beta \in \mathcal{A}^*\left(R^d\right)$, $\beta(i) = s$.*

Proof. Suppose not: there exist $d > 0$ and a site matching $\beta \in \mathcal{A}^*\left(R_i^{s,d}, R_{-i}\right)$ such that $\beta(i) = t \neq s$. Note that $|p^*(R), \alpha|$ is an equilibrium for $R^d := \left(R_i^{s,d}, R_{-i}\right)$. Therefore, $p^*\left(R^d\right) \leq p^*(R)$. Since $t \notin D_S\left(R_i^{s,d}, p^*(R)\right)$, it follows by monotonicity of preferences that $p_t^*\left(R^d\right) < p_t^*(R)$. Since R_i is a negative s-translation of $R_i^{s,d}$, $\left|p^*\left(R^d\right), \beta\right|$ is an equilibrium for R, a contradiction. ■

Lemma 2 was stated first in the body text but is repeated here for convenience.

Lemma 2. *Let $N' \subseteq N$ and let $d := 2\max_{s\in S}\{e_s\}$. Let $(s_j)_{j\in N'}$ be an arbitrary list of sites. There is a site $t \in (s_j)_{j\in N'}$ such that*

$$p_t^*\left(\left(R_j^{s_j,d}\right)_{j\in N'}, R_{N\setminus N'}\right) \geq p_t^*(R).$$

Proof. By Property 2, if $s_i \notin D_S(R_i, p^*(R))$, then there exists $\bar{d}_i$ such that $p^*\left(R_i^{s_i,\bar{d}_i}, R_{-i}\right) = p^*(R)$ and $s_i \in D_S\left(R_i^{s_i,\bar{d}_i}, p^*(R)\right)$. The same holds for each $i \in N'$: $p^*\left(\left(R_j^{s_j,\bar{d}_j}\right)_{i\in N'}, R_{N\setminus N'}\right) = p^*(R)$. Thus we assume without loss of generality that $R = \left(\left(R_j^{s_j,\bar{d}_j}\right)_{i\in N'}, R_{N\setminus N'}\right)$.

Assume $\alpha \in \mathcal{A}^*(R)$ is a balanced site-assignment. Let $R^d := \left(\left(R_j^{s_j,d}\right)_{j\in N'}, R_{N\setminus N'}\right)$, $\hat{S} := \left\{t \in S : p_t^*\left(R^d\right) < p_t^*(R)\right\}$, and $\hat{N} := \left\{i \in N : F^*(R^d)\ P_i\ F^*(R)\right\}$. It is clear that if $t \in \hat{S}$ then $\alpha^{-1}(t) \subseteq \hat{N}$. Therefore

$$\alpha^{-1}\left(\hat{S}\right) \subseteq \hat{N}. \tag{7}$$

If $t \in \hat{S}$ has endowment divisible value at $p^*(R)$, then $c_t\left(p^*(R^d)\right) \leq c_t\left(p^*(R)\right) - 1$. Since α is balanced, $c_t\left(p^*(R)\right) - 1 \leq \left|\alpha^{-1}(t)\right|$ and therefore $c_t\left(p^*(R^d)\right) \leq \left|\alpha^{-1}(t)\right|$. If $t \in \hat{S}$ does not have endowment divisible value, $c_t\left(p^*(R^d)\right) \leq c_t\left(p^*(R)\right) = \left|\alpha^{-1}(t)\right|$. In sum,

$$\forall r \in \hat{S},\ \left|\alpha^{-1}(r)\right| \geq c_r\left(p^*(R^d)\right). \tag{8}$$

To arrive at a contradiction, assume that for each $r \in (s_j)_{j\in N'}$, $p_r^*\left(R^d\right) < p_r^*(R)$. Clearly $N' \subseteq \hat{N}$. We claim that for each $k \in \hat{N}$, $D_S\left(R_k^d, p^*\left(R^d\right)\right) \subseteq \hat{S}$. By construction, for $k \in N'$, $D_S\left(R_k^d, p(R^d)\right) \subseteq (s_j)_{j\in N'} \subseteq \hat{S}$. Let $k \notin N'$. Since $R_k^d = R_k$ and preferences are increasing, $F^*\left(R^d\right)\ P_k\ F^*(R)$ implies the result directly. Let $\beta \in \mathcal{A}^*\left(R^d\right)$. What we have just shown implies $\hat{N} \subseteq \beta^{-1}(\hat{S})$. By 8,

$$\left|\hat{N}\right| \leq \left|\beta^{-1}\left(\hat{S}\right)\right| \leq \sum_{r\in\hat{S}} c_r\left(p^*(R^d)\right) \leq \sum_{r\in\hat{S}} \left|\alpha^{-1}(r)\right| = \left|\alpha^{-1}\left(\hat{S}\right)\right|,$$

where the last equality is because the $\left(\alpha^{-1}(r)\right)_{r\in\hat{S}}$ are disjoint. Combined with line 7, we deduce that $\hat{N} = \alpha^{-1}\left(\hat{S}\right)$.

By Lemma 1, for each $r \in (s_j)_{j\in N'}$ there is a chain

$$\left(x_{i^1}, \alpha(i^1)\right)\ I_{i^1}\ \left(x_{i^2}, \alpha(i^2)\right) \ldots I_{i^n}\ \left(\frac{w}{p_r^*(R)}, r\right)$$

such that $t := \alpha(i^1)$ is exhausted at (x, α). For each $k \notin \hat{N}$, since $k \notin N'$, $R_k^d = R_k$ and $D_S\left(R_k, p^*(R)\right) \cap \hat{S} = \emptyset$. If this were not true then $D_s\left(R_k, p^*\left(R^d\right)\right) \subseteq \hat{S}$ and it would follow that $F^*\left(R^d\right)\ P_k\ F^*(R)$. Therefore, since $r \in \hat{S}$, $i^n \in \hat{N}$. Then $\alpha(i^n) \in \hat{S}$, and it follows by the same argument that $i^{n-1} \in \hat{N}$, and so on. Conclude that $\{i^1, \ldots, i^n\} \subseteq \hat{N}$ and therefore that $t \in \hat{S}$. Since t is exhausted at (x, α), it has endowment divisible value at $p^*(R)$. Moreover, $\left|\alpha^{-1}(t)\right| = c_t\left(p^*(R)\right)$ and $c_t\left(R^d\right) = c_t(R) - 1$. Therefore, $\left|\alpha^{-1}(t)\right| > c_t\left(R^d\right)$.

Thus, $c_t\left(p^*(R^d)\right) = |\alpha^{-1}(t)| - 1$. Therefore, since $\hat{N} = \alpha^{-1}\left(\hat{S}\right)$, line (8) then implies

$$\left|\hat{N}\right| = \left|\alpha^{-1}\left(\hat{S}\right)\right| > \sum_{\hat{s}\in\hat{S}} c_{\hat{s}}\left(p^*(R^d)\right),$$

a contradiction. ■

With Lemma 2 shown, we may now use the fact that F^* is *weakly group-strategy-proof.*

Proof of Property 1. By Property 2, we may confine attention to the case when $s \in D_S(R_i, \pi(0))$. Suppose that for $d > 0$, $\pi_s(d) < \pi_s(0)$. If $s \notin D_S\left(R_i^d, \pi(d)\right)$, then we apply Property 2 to conclude that $\pi_s(0) = \pi_s(d) < \pi_s(0)$. Therefore, $s \in D_S\left(R_i^d, \pi(d)\right)$. By definition of R^d, for each pair of bundles (x,s) and (y,t) with $t \neq s$, if $(x,s)\ I_i^d\ (y,t)$ then $(y,t)\ P_i\ (x,s)$. Therefore, for each $\alpha \in \mathcal{A}^*\left(R^d\right)$,

$$|\pi(d), \alpha(i)|\ P_i\ ({}^{w}/_{\pi_s(d)}, s)\ P_i\ F^*(R),$$

contradicting *strategy-proofness.* ■

Appendix B. Proof of Theorem 5

Proposition 3. *F^* satisfies constant-bundle continuity.*

Proof. Let $f \in F^*$. Let $R^n \in \mathcal{R}^N$ be a sequence converging to $R \in \mathcal{R}^N$. Let (x,α) be an allocation such that for each $n \in \mathbb{N}$, $f(R^n) = (x,\alpha)$. Let $\hat{S} := \alpha(N)$. For each $s \in S \setminus \hat{S}$ and each $n \in \mathbb{N}$, $p_s^*(R^n) = we_s^{-1}$. Clearly, for each $s \in \hat{S}$, $p_s^*(R^n)$ is a constant sequence, so we conclude that $p^*(R^n)$ is a constant sequence: $p^*(R^n) \equiv \bar{p}$. By Lemma 3, there is $\bar{n} \in \mathbb{N}$ such that for each $n > \bar{n}$, and each $i \in N$, $D_S(R_i, \bar{p}) \supseteq D_S(R_i^n, \bar{p})$. This implies moreover that $\bar{p} \in \mathbb{P}(R)$ and therefore, $p^*(R) \leq \bar{p}$.

Let $n \geq \bar{n}$. Let each agent i transition from R_i^n to R_i in three steps. Construct preference relation $\hat{R}_1$ such that $D_S\left(\hat{R}_1, \bar{p}\right) = D_S(R_1, \bar{p})$ by performing successive, positive site-translations. By Property 2 , $p^*\left(\hat{R}_1, R_{-1}^n\right) = \bar{p}$, $p^*\left(\hat{R}_1, \hat{R}_2, R_{N\setminus\{1,2\}}^n\right) = \bar{p}$ and so on to conclude that $p^*\left(\hat{R}\right) = \bar{p}$. Next, for each i, construct $\tilde{R}_i$ from $\hat{R}_i$ by site translations such that $(x,s)\ \tilde{R}_i\ D\left(\tilde{R}_i, \bar{p}\right)$ if and only if $(x,s)\ R_i\ D(R_1, \bar{p})$. That is, the optimizing indifference set of $\tilde{R}_i$ for prices $\bar{p}$ is identical to the optimizing indifference set of R_i for prices $\bar{p}$. Since we have already set $D_S\left(\hat{R}_i, \bar{p}\right) = D_S\left(R_i, \bar{p}\right)$, this operation involves sites $s \notin D_S\left(\hat{R}_i, \bar{p}\right)$. Therefore, by Property 2 again conclude that $p^*\left(\tilde{R}\right) = \bar{p}$. By *strategy-proofness*, $f\left(R_1, \tilde{R}_{-1}\right)\ R_1\ f\left(\tilde{R}\right)$. If $f\left(R_1, \tilde{R}_{-1}\right)\ P_1\ f\left(\tilde{R}\right)$, then by construction $f\left(R_1, \tilde{R}_{-1}\right)\ \tilde{P}_1\ f\left(\tilde{R}\right)$,contradicting *strategy-proofness*. Therefore, $f\left(R_1, \tilde{R}_{-1}\right)\ I_1\ f\left(\tilde{R}\right)$. It follows that, since the preferences of other agents remain constant, $f\left(R_1, \tilde{R}_{-1}\right)$ is an equilibrium for $\tilde{R}$ and $\bar{p} = p^*\left(\tilde{R}\right) \leq p^*\left(R_1, \tilde{R}_{-1}\right)$. We conclude then that $p^*\left(R_1, \tilde{R}_{-1}\right) = \bar{p}$. Proceed inductively to conclude that $p^*\left(R\right) = p^*\left(\tilde{R}\right) = \bar{p}$. ■

B.1. **The domain of unique assignment cardinality.** We define sub-domain $\mathcal{D}^* \subset \mathcal{R}^N$, which we call the **domain of unique assignment cardinality** or the **domain of unique size** for short. This domain contains all preference profiles $R \in \mathcal{R}^N$ with the property that if α and $\beta \in \mathcal{A}^*(R)$, then for each $s \in S$, $|\alpha^{-1}(s)| = |\beta^{-1}(s)|$.

Proposition 4. *For each $R \in \mathcal{R}^N \setminus \mathcal{D}^*$, there is a sequence $R^n \in \mathcal{D}^*$ such that R^n converges to R, and $p^*(R^n)$ converges to $p^*(R)$ from below.*

Proof. As before we normalize $e = \mathbf{1}$ and $w = 1$.

Let $\alpha \in \mathcal{A}^*(R)$ be balanced for $p^*(R)$. Let

$$\hat{S} := \left\{s \in S : \left|\alpha^{-1}(s)\right| = c_s\left(p^*(R)\right) - 1\right\}.$$

Define the mapping $\varepsilon \in \mathbb{R} \mapsto p^\varepsilon \in \mathbb{R}^S$ for each $s \in S$ as follows:

$$p_s^\varepsilon := \begin{cases} p_s^*\left(R\right) - \frac{\varepsilon\left[p_s^*(R)\right]^2}{1+\varepsilon p_s^*(R)} & s \in \hat{S} \\ p_s^*(R) & s \in S \setminus \hat{S}. \end{cases}$$

Similarly, define the mapping $\varepsilon \mapsto R^\varepsilon \in \mathcal{R}^N$ by setting, for each $i \in N$, $R_i^\varepsilon := R_i^{\hat{S}, -\varepsilon}$.

By construction, $|p^\varepsilon, \alpha|$ is an equilibrium for R^ε and therefore $p^*\left(R^\varepsilon\right) \leq p^\varepsilon$. Recall that, given our normalization, $c\left(p\right) = \lfloor p \rfloor$ (where the *floor* operation is calculated component-wise). Therefore, the sites $s \in \hat{S}$ are precisely those satisfying $|\alpha^{-1}(s)| = \lfloor p_s^*\left(R\right) \rfloor - 1$. Thus, since α is balanced,

$$\sum_{s \in S} \lfloor p_s^*\left(R\right) \rfloor = |N| + \left|\hat{S}\right|.$$

But now for each $s \in \hat{S}$, $\lfloor p_s^{\varepsilon} \rfloor = \lfloor p_s^* (R) \rfloor - 1$. Therefore

$$\sum_{s \in S} \lfloor p_s^{\varepsilon} \rfloor = |N|, \tag{9}$$

which further implies that $R^{\varepsilon} \in \mathcal{D}$.

Let ε^n be a sequence of positive numbers converging to zero. For each $n \in \mathbb{N}$, let $p^n := p^{\varepsilon^n}$ and $R^n := R^{\varepsilon^n}$. Clearly, p^n converges to p from below, and for each $n \in \mathbb{N}$, $p^*(R^n) \leq p^n$. Thus $p^*(R^n)$ is bounded above by p. Since R^n converges to R and p^* is lower semi-continuous (Corollary 1), $\liminf p^*(R^n) \geq p^*(R)$. In sum,

$$\liminf p^* (R^n) \geq p^* (R) = \lim p^n \geq \limsup p^* (R^n),$$

implying that $\lim p^*(R^n)$ exists and $\lim p^* (R^n) = p^*(R)$. ■

Proposition 5. $\mathcal{R}^N \setminus \mathcal{D}^*$ *is closed.*

Proof. Normalize $w = 1$ and $e \equiv 1$. For each $R \in \mathcal{R}^N$, let

$$U(R) := \{s \in S : \ s \notin \cup_{i \in N} D(R_i, p^*(R))\}$$

be the **undesirable sites**. Note first that $R \in \mathcal{R}^N \setminus \mathcal{D}^*$ if and only if

$$\sum_{s \in S} \left\lfloor \frac{1}{p_s^*(R)} \right\rfloor > |N| + |U(R)|. \tag{10}$$

Now let R^n be a sequence in $\mathcal{R}^N \setminus \mathcal{D}^*$ converging to R. The lower semi-continuity of $p^*(R)$ implies that for each $s \in S$,

$$\frac{1}{p_s^*(R)} \geq \frac{1}{\liminf p_s^*(R^n)}.$$

Moreover, $|U(R)| = \lim |U(R^n)|$. For clarity of exposition we neglect this term in the following, but it will be clear that it does not change the result. Since each $R^n \in \mathcal{R}^N \setminus \mathcal{D}^*$, for each $\varepsilon > 0$, there exists $\bar{n} \in \mathbb{N}$ such that for each $n > \bar{n}$,

$$\sum_{s \in S} \frac{1}{p_s^*(R) - \varepsilon} \geq \sum_{s \in S} \frac{1}{\liminf p_s^*(R^n) - \varepsilon} > \sum_{s \in S} \frac{1}{p_s^*(R^n)} \geq \sum_{s \in S} \left\lfloor \frac{1}{p_s^*(R^n)} \right\rfloor \geq |N| + 1,$$

and the result follows. ■

A single-valued rule φ is **individually invariant to unilateral positive translation**, or **unilaterally invariant** for short, if for each $R \in \mathcal{R}^N$, and each $i \in N$, if $(x_i, s_i) = \varphi_i(R)$, then for each $d > 0$, $(x_i, s_i) = \varphi_i\left(R_i^{s,d}, R_{-i}\right)$. There is a well-known result in the literature, called the "Invariance Lemma" by Thomson (2014), that implies each *strategy-proof* rule is unilaterally invariant. In fact, the invariance holds for more general changes in preferences. *Unilateral invariance*, while being implied by *strategy-proofness*, is in fact closely related to *strategy-proofness*. See Klaus and Bochet (2013) for a thorough study.

Let $R \in \mathcal{D}^*$ and $p := p^*(R)$. Let $\bar{c} \in \mathbb{N}^S$ be the list of numbers such that $|p, \alpha| \in F^*(R)$ implies for each site that $|\alpha^{-1}(s)| = \bar{c}_s$. It is without loss of generality to assume that $\bar{c} > 0$ as undesirable sites remain undesirable in our arguments. We argued above that $R \in \mathcal{R} \setminus \mathcal{D}^*$ if and only if inequalty 10 holds. Therefore, $R \in \mathcal{D}^*$ if and only if its negation holds. Since we have assumed no undesirable sites, and since feasibility requires that $|N|$ not exceed the total available capacity, we deduce that $R \in \mathcal{D}^*$ if and only if

$$\sum_{s \in S} \left\lfloor \frac{1}{p_s^*(R)} \right\rfloor = |N|. \tag{11}$$

Let R_0^p be a preference relation such that B, the budget set given by prices p, is an indifference set of R_0^p. That is, given prices p, an agent with preferences R_0^p is indifferent as to which site's commodity he wants to consume. Let R^p be the profile such that for each agent i, $R_i^p = R_0^p$. By repeated applications of Property 2, conclude that $p^*(R^p) = p^*(R) = p$ and $F^*(R^p) \supseteq F^*(R)$. *Welfare anonymity* implies that all agents consume on the same indifference set. Together with *strong-undomination* we deduce that $\varphi(R^p) = |p, \beta|$ for some $\beta \in S^N$, and therefore that $\varphi(R^p) \in F^*(R^p)$.

Let $N^* := \{i \in N : \exists \alpha \in \mathcal{A}^*(R),\ \alpha(i) \text{ exhausted at } |p, \alpha|\}$.

Lemma. *Let $i \in N$ and $\hat{R} := (R_{-i}, R_i^p)$. For each $j \in N$, $\varphi_j\left(\hat{R}\right) \in D(\hat{R}_j, p)$.*

Proof. Let $\varepsilon > 0$. Let $(x, \alpha) \in F^*(R)$ and let $i^* \in N^*$ be such that $\alpha(i^*)$ is exhuasted at (x, α). We first show by induction that the lemma is true for profile $\left((R_j^{\alpha(j),\varepsilon})_{j \neq i^*}, R_{i^*}^p\right)$.
The Inductive Base: Let $\varphi(R^p) = |p, \beta|$. For each $i \in N \setminus i^*$, there is a $j \in \beta^{-1}(\alpha(i))$, implying that $\varphi_j(R^p) = (x_i, \alpha(i))$. Let $\bar{R}_j := R_i^{\alpha(i),\varepsilon}$ and let $(z, \gamma) = \varphi\left(\bar{R}_j, R_{-j}^p\right)$. By *unilateral invariance*, $(z_j, \gamma(j)) = (x_i, \alpha(i))$. For each $k \in N \setminus j$, $F^*\left(\bar{R}_j, R_{-j}^p\right) I_k^p B$, therefore by *strong undomination*, there exists $\bar{k} \in N \setminus j$ such that $\varphi_{\bar{k}}(\bar{R}_j R_{-j}^p)\ R_{\bar{k}}^p\ B$. By *welfare anonymity*, for each $k \in N \setminus j$, $(z_k, \gamma(k))\ I_k^p\ \left(z_{\bar{k}}, \gamma(\bar{k})\right)$. Therefore, for each $k \in N \setminus j$, $\varphi_k(\bar{R}_j R_{-j}^p)\ R_k^p\ B$. We deduce via equation 11 that for each $s \in S$, $|\gamma^{-1}(s)| = |\alpha^{-1}(s)|$. Since

$i \neq i^*$, this further implies there is $\hat{k} \in N \setminus j$ satisfying $\gamma(\hat{k}) = \alpha(i^*)$ and, since preferences are increasing, $z_{\hat{k}} \geq x_{i^*}$. This also holds for each $k' \in \gamma^{-1}(\alpha(i^*))$ and so, by feasibility, $z_{\hat{k}} = x_{i^*}$ (recall that $\alpha(i^*)$ is exhausted at (x, α)). Since $\hat{k}$'s preferences are R_0^p, *welfare anonymity* implies that for each $k' \in N \setminus j$, $\varphi_{k'}\left(\bar{R}_j, R_{-j}^p\right) \in B$.

By *welfare anonymity*,

$$\varphi_i\left(R_i^{\alpha(i),\varepsilon}, R_{-i}^p\right) I_i^{\alpha(i),\varepsilon} \varphi_j\left(\bar{R}_j, R_{-j}^p\right).$$

Finally, letting $\bar{R} := \left(R_i^{\alpha(i),\varepsilon}, R_{-i}^p\right)$, *strong undomination* imples that for each $i \in N$, $\varphi_i\left(\bar{R}\right) \in D\left(\bar{R}_i, p\right)$, as desired.

The Induction Step: Fix $n \in \mathbb{N}$. The induction hypothesis is as follows: Let $\hat{R} \in \mathcal{R}^N$ and $N' := \left\{i \in N : \hat{R}_i \neq R_0^p\right\}$. Assume that $|N'| \leq n$ and, for each $i \in N'$, there is $j \in N \setminus i^*$ such that $\hat{R}_i = R_j^{\alpha(j),\varepsilon}$. Then, for each $i \in N$, $\varphi_i\left(\hat{R}\right) \in D\left(\hat{R}_i, p\right) \subseteq B$.

Let $\left((R_i^{\alpha(i),\varepsilon})_{i\in N'}, R_{N\setminus N'}^p\right)$ satisfy the induction hypothesis and let $\varphi\left((R_i^{\alpha(i),\varepsilon})_{i\in N'}, R_{N\setminus N'}^p\right) = (y, \beta)$. For compact notation, let $R_{N'}^{\alpha,\varepsilon}$ denote the partial profile $(R_i^{\alpha(i),\varepsilon})_{i\in N'}$.

Claim 2. For each $i \in N \setminus N'$, there is $j \in N \setminus N'$ such that $\beta(j) = \alpha(i)$.

Proof. Let $\alpha(i) = s$. The claim is thus

$$i \in N \setminus N' \implies (\exists j \in N \setminus N' \; s.t. \; \beta(j) = s).$$

We show the contrapositive. The induction hypothesis and equation 11 imply that, for each site $s' \in S$, $|\beta^{-1}(s')| = \bar{c}_{s'}$. For each $k \in N'$, since $(y_k, \beta(k)) \in D\left(R_k^{\alpha(k),\varepsilon}, p\right)$, then $(y_k, \beta(k)) = (x_i, \alpha(i))$. Therefore if $\beta^{-1}(s) \subset N'$, then $|\alpha^{-1}(s) \cap N'| = \bar{c}_s$. Since $\bar{c}_s = |\alpha^{-1}(s)|$ by definition, this further implies $\alpha^{-1}(s) \subseteq N'$ and $i \in N'$. □

Let $i \in N \setminus (N' \cup i^*)$. By Claim 2, there exists $j \in N \setminus N'$ such that $\beta(j) = \alpha(i)$. Let $\bar{R}_j := R_i^{\alpha(i),\varepsilon}$, and denote

$$\bar{R} := \left(\bar{R}_j, R_{N'}^{\alpha,\varepsilon}, R_{N\setminus(N'\cup j)}^p\right).$$

Let $\varphi\left(\bar{R}\right) = (z, \gamma)$. *Unilateral invariance* implies that $(z_j, \gamma(j)) = (x_i, \alpha(i))$.

Claim 3. For each $k \in N$,

$$(z_k, \gamma(k)) \; \bar{R}_k \; D\left(\bar{R}_k, p\right)$$

Proof. The proof is by contradiction. Assume there are $k \in N$ and $(\bar{z}, r) \in D\left(\bar{R}_k, p\right)$ such that

$$(\bar{z}, r)\ \bar{P}_k\ (z_k, \gamma(k))\,.$$

Assume first that $k \in N \setminus (N' \cup j)$. Thus, k's preferences are R_0^p. Since for each $\bar{k} \in N$ and each $f \in F^*$, $f(\bar{R})\ \bar{I}_{\bar{k}}\ D\left(\bar{R}_k, p\right)$, *strong undomination* implies there is $k' \in N$ for whom

$$(z_{k'}, \gamma(k'))\ \bar{P}_{k'}\ D(\bar{R}_{k'}, p),$$

which further implies, since preferences are increasing, that $z_{k'} > wp^{-1}_{\gamma(k')}$. *Welfare anonymity* implies that the preferences of k and k' differ and therefore that $k' \in N' \cup j$. Profile $\bar{\bar{R}} := \left(R^p_{k'}, \bar{R}_{-k'}\right)$ satisfies the induction hypothesis and therefore, for each $i \in N$, $\varphi_i\left(\bar{\bar{R}}\right) \in D\left(\bar{\bar{R}}_i, p\right) \subset B$. Then if $\bar{\bar{R}}$ is the true profile, k' will manipulate by reporting $\bar{R}_{k'}$, contradicting *strategy-proofness*. Conclude that $k \notin N \setminus (N' \cup j)$.

Assume that $k \in N'$; k's preferences are $R_k^{\alpha(k),\varepsilon}$. Profile $\left(R^p_k, \bar{R}_{-k}\right)$ satisfies the induction hypothesis and therefore $\varphi\left(R^p_k, \bar{R}_{-k}\right) \in B^N$. We also apply Claim 2: there is $k' \in N$ with preferences R^p_0 such that $\varphi_{k'}\left(R^p_k, \bar{R}_{-k}\right) = (x_k, \alpha(k))$. Let $\bar{\bar{R}}_{k'} := R_k^{\alpha(k),\varepsilon}$ and denote $\bar{\bar{R}} := \left(\bar{\bar{R}}_{k'}, R^p_k, \bar{R}_{N\setminus\{k,k'\}}\right)$. By *unilateral invariance*, $\varphi_{k'}\left(\bar{\bar{R}}\right) = (x_k, \alpha(k))$. By *welfare anonymity*,

$$\varphi_k\left(\bar{R}\right)\ I_k^{\alpha(k),\varepsilon}\ \varphi_{k'}\left(\bar{\bar{R}}\right)\ R_k^{\alpha(k),\varepsilon}\ D(\bar{R}_k, p).$$

This is another contradiction and we conclude $k \notin N' \cup j$. In sum, we have deduced that $k \notin N$, the contradiction we sought. □

Let $k \in N'$. If $\varphi_k\left(\bar{R}\right)\ \bar{P}_k\ D\left(\bar{R}_k, p\right)$, then by the induction hypothesis, at profile $\left(R^p_k, \bar{R}_{-i}\right)$, agent k successfuly manipulates the rule by reporting $\bar{R}_k$. Therefore, for each $k \in N'$, $\varphi_k\left(\bar{R}\right) \in D\left(\bar{R}_i, p\right)$. *Strong undomination* together with *welfare anonymity* imply that for each $\bar{k} \in N \setminus N'$, $\varphi_{\bar{k}}\left(\bar{R}\right)\ \bar{R}_{\bar{k}}\ B$. Equation 11 then implies that, for each $t \in S$, $|\gamma^{-1}(t)| = |\alpha^{-1}(t)|$. Since for each $k \in N'$, $\bar{R}_k \neq R_{i^*}^{\alpha(i^*),\varepsilon}$, as in the base case there is an agent $k \in N \setminus N'$ such that $\gamma(k) = \alpha(i^*)$. Therefore, by feasibility, $\varphi_k\left(\bar{R}\right) = |p, \gamma(k)|$ and it follows from *welfare anonymity* that, for each $i \in N \setminus N'$, $\varphi_i\left(\bar{R}\right) \in B$. Finally, switch the preference relations of i and j and invoke *welfare anonymity* to conclude $\varphi\left((R_j^{\alpha(j),\varepsilon})_{j\in N'\cup i}, R^p_{N\setminus(N'\cup i)}\right) \in F^*\left((R_j^{\alpha(j),\varepsilon})_{j\in N'\cup i}, R^p_{N\setminus(N'\cup i)}\right)$.

Now let $i \in N$ be arbitrary. If $i \in N^*$, then there exists $\alpha \in \mathcal{A}^*(R)$ such that our argument goes through: $\varphi\left(R_i^p, R_{-i}^{\alpha,\varepsilon}\right) \in B^N$ and for each $j \neq i$, $\varphi_j\left(R_i^p, R_{-i}^{\alpha,\varepsilon}\right) \in D(R_j^{\alpha(j),\varepsilon}, p)$. If $i \notin N^*$, then there exists $i^0 \in N^*$, $\alpha \in \mathcal{A}^*(R)$, and an indifference chain

$$\left|p, \alpha(i^0)\right| I_{i^0} \left|p, \alpha(i^1)\right| I_{i^1} \cdots I_{i^{k-1}} \left|p, \alpha(i)\right|$$

such that $\alpha(i^0)$ is exhausted at $|p, \alpha|$. Now construct site-assignment α^* as follows, letting $i^k = i$:

$$\alpha^*(j) := \begin{cases} \alpha\left(i^{l+1}\right) & j = i^l \mod k+1 \\ \alpha(j) & otherwise. \end{cases}$$

Note that $p^*(R_i^p, R_{-i}) = p^*(R)$ and $\alpha^* \in \mathcal{A}^*(R_i^p, R_{-i})$. Therefore, our argument holds for profile (R_i^p, R_{-i}) by setting $i^* = i$ and using site assignment α^*. Conclude that for each $j \neq i$, $\varphi_j\left(R_i^p, R_{-i}^{\alpha^*,\varepsilon}\right) \in D(R_j, p)$ and $\varphi_i\left(R_i^p, R_{-i}^{\alpha^*,\varepsilon}\right) \in D(R_i^p, p)$.

We have shown that for each $i \in N$, each site assignment $\alpha \in S^N$ satisfying either $\alpha \in \mathcal{A}^*(R)$ or α is constructed as α^*, each $\varepsilon > 0$, and each $j \neq i$, $\varphi_j^{i,\varepsilon} := \varphi_j\left(\left(R_j^{\alpha(j),\varepsilon}\right)_{j\neq i}, R_i^p\right) \in D\left(R_j^{\alpha(j),\varepsilon}, p\right)$. Moreover, $\varphi_i^{i,\varepsilon} \in D(R_i^p, p)$. Therefore, there exists $\beta \in S^N$ such that $\varphi^{i,\varepsilon} = |p, \beta|$. By construction, for each $j \neq i$, $\varphi_j^{i,\varepsilon} = |p, \alpha(j)|$. Recall that $\sum_{s\in S} \bar{c}_s(p) = |N|$, which leaves $\varphi_i^{i,\varepsilon} = |p, \alpha(i)|$. Since $\varepsilon > 0$ was arbitrary, *constant sequence continuity* implies that $\varphi_i(R_{-i}, R_i^p) \in D(R_i^p, p)$ and, for each $j \neq i$, $\varphi_j(R_{-i}, R_i^p) \in D(R_j, p)$. ■

It remains only to show that at $\varphi(R)$, all agents are maximizing their R preferences on B. If there is an agent k with

$$\varphi_k(R)\ P_k\ D(R_k, p),$$

then since preferences are increasing, $\varphi_k(R)$ is above B. When (R_k^p, R_{-k}) is the true profile, k manipulates by reporting R_k, a contradiction. Thus we have that for each agent i,

$$D(R_i, p)\ R_i\ \varphi_i(R).$$

But then by *strong undomination* we have for each agent i that $\varphi_i(R)\ R_i\ D(R_i, p)$, and the proof is complete. ■

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