TIME-CONSISTENT POLICY

A Project

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MOTIVATION

- In many many policy situations the commitment solution is time-inconsistent.
- The problem occurs in finite- and infinite-horizon contexts. The continuous Markov solution is a natural equilibrium to study, as it is the connection between these two kinds of setups.
- Existing methods for finding these equilibria are
  - a little of the black box type, and therefore not easily accessible to a large audience and
  - based on, in some sense, arbitrary approximations (in particular, steady states are not pinned down in these models with controlled accuracy).
- We study a particular example to illustrate
  - how to make the economics of the problem palatable and
  - describe a computational algorithm that allows controlled accuracy.

The economics of the problem comes out via the GEE. This equation, which is the key one for any policy maker, can be stated explicitly. It can also be evaluated at a calibrated point and hence both the qualitative and the quantitative strengths of the different costs and benefits for raising taxes can be studied. The GEE focuses on the "natural", or "basic" economics, in the sense that it is the same economics as shows up in any finite-horizon model: the incentives for raising and lowering policy are the pure economic ones, entirely ignoring "reputational" concerns that could take a variety of forms and are therefore hard to fully describe.

Recursive equilibrium for expenditures given by the (exogenous) function $g = \Psi(k)$: $k' = H(k)$ and $l = L(k)$, satisfying the FOC for labor

$$\frac{u_l\left(f(k, L(k)) + (1-\delta)k - H(k) - G(k), 1 - L(k), G(k)\right)}{u_c\left(f(k, L(k)) + (1-\delta)k - H(k) - G(k), 1 - L(k), G(k)\right)} = f_l(k, L(k))(1 - \Psi(k)),$$

where

$$G(k) \equiv \Psi(k)\left(f(k, L(k)) - \delta k\right),$$

and the FOC for saving

$$\begin{aligned} u_c\left(f(k, L(k)) + (1-\delta)k - H(k) - G(k), 1 - L(k), G(k)\right) = \\ \beta\left(1 + (f_k(H(k), L(H(k))) - \delta)(1 - \Psi(H(k)))\right) \cdot \\ u_c\left(f(H(k), L(H(k))) + (1-\delta)H(k) - H(H(k)) - G(H(k)), 1 - L(H(k)), G(H(k))\right). \end{aligned}$$

One-period deviation in expenditures from the above equilibrium: current expenditures are given by an arbitrary τ and future policies given by $\Psi(\cdot)$. That boils down to current equilibrium functions $k' = H(k,\tau)$ and $l = L(k,\tau)$ and future functions as above. The FOC for labor in the current period reads

$$\frac{u_l\left(f(k, L(k,\tau)) + (1-\delta)k - H(k,\tau) - G(k,\tau), 1 - L(k,\tau), G(k,\tau)\right)}{u_c\left(f(k, L(k,\tau)) + (1-\delta)k - H(k,\tau) - G(k,\tau), 1 - L(k,\tau), G(k,\tau)\right)} = f_l(k, L(k,\tau))(1-\tau),$$

where

$$G(k,\tau) \equiv \tau\left(f(k, L(k,\tau)) - \delta k\right),$$

and the FOC for current saving reads

$$\begin{aligned} u_c\left(f(k, L(k,\tau)) + (1-\delta)k - H(k,\tau) - G(k,\tau), 1 - L(k,\tau), G(k,\tau)\right) = \\ \beta\left(1 + (f'(H(k,\tau)) - \delta)(1 - \Psi(H(k,\tau)))\right) \cdot \\ u_c\left(f(H(k,\tau), L(H(k,\tau))) + (1-\delta)H(k,\tau) - H(H(k,\tau)) - G(H(k,\tau)), 1 - L(H(k,\tau)), G(H(k,\tau))\right). \end{aligned}$$

Notice: the functions are defined so that

$$\begin{aligned} H(k, \Psi(k)) &= H(k) \\ L(k, \Psi(k)) &= L(k) \\ G(k, \Psi(k)) &= G(k). \end{aligned}$$

The government decision:

$$\max_{\tau} u\left(f(k, L(k,\tau)) + (1-\delta)k - H(k,\tau) - G(k,\tau), 1 - L(k,\tau), G(k,\tau)\right) + \beta V(H(k,\tau))$$

where

$$V(k) = u\left(f(k, L(k,\Psi(k))) + (1-\delta)k - H(k,\Psi(k)) - G(k,\Psi(k)), 1 - L(k,\Psi(k)), G(k,\Psi(k))\right) + \beta V(H(k,\Psi(k)))$$

for all k.

Notice that, on the equilibrium path, when it is optimal for the government to choose $\Psi(k)$ for every k, the max in the government's problem equals $V(k)$! That is, we can specify the government's problem as a recursive one.

Procedure:

- Taking the first-order conditions with respect to τ, we obtain an equation where $V'(H(k, \Psi(k)))$ needs to be found.
- One then differentiates the expression for $V(k)$. This will result in a number of terms, some of which multiply $\Psi'(k)$; all those terms cancel, using the first-order condition just derived (the envelope theorem being at work).
- Another term remains in this expression: $\beta V'(H(k, \Psi(k)))H_k(k, \Psi(k))$. But it can be solved for from the first-order condition, delivering an expression for $V'(k)$ not involving any further value function derivative.
- Evaluate the expression for V' at $H(k, \Psi(k))$ and substitute it into the first-order condition. This is now the GEE.

What we obtain is:

$$u_c[L_\tau f_l - H_\tau - G_\tau] - u_l L_\tau + u_g G_\tau + \beta H_\tau \cdot$$
$$\left\{ u_c'[f_k' + f_l' L_k' + 1 - \delta - H_k' - G_k'] - u_l' L_k' + u_g' G_k' - \frac{H_k'}{H_\tau'} (u_c'[f_l' L_\tau' - H_\tau' - G_\tau'] - u_l' L_\tau' + u_\tau' G_\tau') \right\} = 0.$$

There are several parts here: the effects of an increase in the tax rate on:

1. Current consumption,
   (a) via higher labor supply ($L_\tau f_l$);
   (b) via lower savings (H_τ); and
   (c) via higher government spending (G_τ).
2. Current leisure, via higher labor supply (L_τ).
3. Current government spending, which goes up (G_τ).
4. Future utility-relevant variables, via a decrease in savings (H_τ), leading to:
   (a) a change in next period's consumption via a direct effect on production and undepreciated capital ($f_k' + 1 - \delta$), an indirect effect on labor supply ($f_l' L_k'$), an indirect effect on saving ($-H_k'$), and an indirect on government spending (G_k');

(b) a change in next period's leisure, through labor supply $(-L'_k)$; and

(c) a change in next period's government spending directly (G'_k).

(d) an induced change in taxes $(-H'_k/H'_\tau)$, which affects

  i. next period's consumption, via the same channels as in the initial period $(f'_l L'_\tau - H'_\tau - G'_\tau)$;

  ii. next period's leisure (L'_τ); and

  iii. next period's government spending (G'_τ).

Lump-sum taxes

The analysis is quite parallel to the one above. Recursive equilibrium for expenditures given by the (exogenous) function $\tau = \Psi(k)$: $k' = H(k)$ and $l = L(k)$, satisfying the FOC for labor

$$\frac{u_l\left(f(k, L(k)) + (1-\delta)k - H(k) - \Psi(k), 1 - L(k), \Psi(k)\right)}{u_c\left(f(k, L(k)) + (1-\delta)k - H(k) - \Psi(k), 1 - L(k), \Psi(k)\right)} = f_l(k, L(k))$$

and the FOC for saving

$$\begin{aligned} u_c\left(f(k, L(k)) + (1-\delta)k - H(k) - \Psi(k), 1 - L(k), \Psi(k)\right) = \\ \beta\left(1 - \delta + f_k(H(k), L(H(k)))\right) \cdot \\ u_c\left(f(H(k), L(H(k))) + (1-\delta)H(k) - H(H(k)) - \Psi(H(k)), 1 - L(H(k)), \Psi(H(k))\right). \end{aligned}$$

Notice, compared to before that

- the current expenditure function, $G(k)$, has to equal $\Psi(k)$ and therefore does not have to be included in the analysis; and
- the FOCs both change due to the absence of a distortion.

Just like before, define a one-period deviation equilibrium with current expenditures given by an arbitrary τ and future policies given by $\Psi(\cdot)$. The current equilibrium functions are $k' = H(k,\tau)$ and $l = L(k,\tau)$ and the future functions are as above. The FOC for labor in the current period reads

$$\frac{u_l\left(f(k, L(k,\tau)) + (1-\delta)k - H(k,\tau) - \tau, 1 - L(k,\tau), \tau\right)}{u_c\left(f(k, L(k,\tau)) + (1-\delta)k - H(k,\tau) - \tau, 1 - L(k,\tau), \tau\right)} = f_l(k, L(k,\tau)),$$

and the FOC for current saving reads

$$\begin{aligned} &u_c\left(f(k, L(k,\tau)) + (1-\delta)k - H(k,\tau) - \tau, 1 - L(k,\tau), \tau\right) = \\ &\qquad\qquad \beta\left(1 - \delta + f_k(H(k,\tau), L(H(k,\tau)))\right) \cdot \\ &u_c\left(f(H(k,\tau), L(H(k,\tau))) + (1-\delta)H(k,\tau) - H(H(k,\tau)) - \Psi(H(k,\tau)), 1 - L(H(k,\tau)), \Psi(H(k,\tau))\right). \end{aligned}$$

Notice: the functions are defined so that

$$H(k, \Psi(k)) = H(k)$$
$$L(k, \Psi(k)) = L(k).$$

The government decision is now:

$$\max_{\tau} u\left(f(k, L(k,\tau)) + (1-\delta)k - H(k,\tau) - \tau, 1 - L(k,\tau), \tau\right) + \beta V(H(k,\tau))$$

where for all k

$V(k) = u\left(f(k, L(k,\Psi(k))) + (1-\delta)k - H(k,\Psi(k)) - \Psi(k), 1 - L(k,\Psi(k)), \Psi(k)\right) + \beta V(H(k,\Psi(k)))$.

Solving for the GEE, one obtains

$$u_c[L_\tau f_l - H_\tau - 1] - u_l L_\tau + u_g + \beta H_\tau \cdot \left\{ u_c'[f_k' + f_l' L_k' + 1 - \delta - H_k'] - u_l' L_k' - \frac{H_k'}{H_\tau'}\left(u_c'[f_l' L_\tau' - H_\tau' - 1] - u_l' L_\tau' + u_g' G_\tau'\right)\right\} = 0.$$

Here, there are many simplifications: one can use the FOCs above to cancel terms, ending up with

$$u_g - u_c + \beta H_\tau \frac{H_k'}{H_\tau'}(u_g' - u_c') = 0.$$

It is clear that a $\Psi(k)$ such that

$$\frac{u_g\left(f(k, L(k)) + (1-\delta)k - H(k) - \Psi(k), 1 - L(k), \Psi(k)\right)}{u_c\left(f(k, L(k)) + (1-\delta)k - H(k) - \Psi(k), 1 - L(k), \Psi(k)\right)} = 1$$

satisfies this condition for all k. This means that we can combine this equation with the first two FOCs for the labor-leisure and consumption-savings choices (those that are NOT off the equilibrium path) to have three functional equations in three unknown functions H, L, and Ψ. This system is entirely standard and does not involve derivatives of policy functions.

Labor taxes

The recursive equilibrium for expenditures is given by the (exogenous) functions $g = \Psi(k)$, $k' = H(k)$, and $l = L(k)$, satisfying the FOC for labor

$$\frac{u_l\left(f(k, L(k)) + (1-\delta)k - H(k) - G(k), 1 - L(k), G(k)\right)}{u_c\left(f(k, L(k)) + (1-\delta)k - H(k) - G(k), 1 - L(k), G(k)\right)} = f_l(k, L(k))(1 - \Psi(k)),$$

where

$$G(k) \equiv \Psi(k) f_l(k, L(k)) L(k),$$

and the FOC for saving

$$\begin{aligned} u_c\left(f(k, L(k)) + (1-\delta)k - H(k) - G(k), 1 - L(k), G(k)\right) = \\ \beta\left(1 - \delta + f_k(H(k), L(H(k)))\right) \cdot \\ u_c\left(f(H(k), L(H(k))) + (1-\delta)H(k) - H(H(k)) - G(H(k)), 1 - L(H(k)), G(H(k))\right). \end{aligned}$$

Notice that

- we again need to define the function G; and that
- the FOC for savings is undistorted but the one for labor is not.

In the one-period deviation, as in the other cases, the current expenditures are given by an arbitrary τ and future policies given by $\Psi(\cdot)$ so that we have current equilibrium functions $k' = H(k,\tau)$ and $l = L(k,\tau)$ and future functions as above. The FOC for labor in the current period reads

$$\frac{u_l\left(f(k,L(k,\tau)) + (1-\delta)k - H(k,\tau) - G(k,\tau), 1 - L(k,\tau), G(k,\tau)\right)}{u_c\left(f(k,L(k,\tau)) + (1-\delta)k - H(k,\tau) - G(k,\tau), 1 - L(k,\tau), G(k,\tau)\right)} = f_l(k, L(k,\tau))(1-\tau),$$

where

$$G(k,\tau) \equiv \tau f_l(k, L(k,\tau))L(k,\tau),$$

and the FOC for current saving reads

$$\begin{aligned} u_c\left(f(k,L(k,\tau)) + (1-\delta)k - H(k,\tau) - G(k,\tau), 1 - L(k,\tau), G(k,\tau)\right) = \\ \beta\left(1 - \delta + f'(H(k,\tau))\right) \cdot \\ u_c\left(f(H(k,\tau), L(H(k,\tau))) + (1-\delta)H(k,\tau) - H(H(k,\tau)) - G(H(k,\tau)), 1 - L(H(k,\tau)), G(H(k,\tau))\right). \end{aligned}$$

As in the case of income taxation, the functions are defined so that

$$\begin{aligned} H(k, \Psi(k)) &= H(k) \\ L(k, \Psi(k)) &= L(k) \\ G(k, \Psi(k)) &= G(k). \end{aligned}$$

The government decision is as before:

$$\max_{\tau} u\left(f(k, L(k,\tau)) + (1-\delta)k - H(k,\tau) - G(k,\tau), 1 - L(k,\tau), G(k,\tau)\right) + \beta V(H(k,\tau))$$

where for all k

$$V(k) = u\left(f(k, L(k,\Psi(k))) + (1-\delta)k - H(k,\Psi(k)) - G(k,\Psi(k)), 1 - L(k,\Psi(k)), G(k,\Psi(k))\right) + \beta V(H(k,\Psi(k))).$$

What we obtain this time is:

$$u_c[L_\tau f_l - H_\tau - G_\tau] - u_l L_\tau + u_g G_\tau + \beta H_\tau \cdot$$
$$\left\{ u_c'[f_k' + f_l' L_k' + 1 - \delta - H_k' - G_k'] - u_l' L_k' + u_g' G_k - \frac{H_k'}{H_\tau'} (u_c'[f_l' L_\tau' - H_\tau' - G_\tau'] - u_l' L_\tau' + u_\tau' G_\tau') \right\} = 0.$$

This equation is identical to the one under income taxation. Using FOCs will make some terms cancel, however: those involving the savings choice. We obtain:

$$L_\tau[u_c f_l - u_l] + G_\tau[u_g - u_c] + \beta H_\tau \cdot$$
$$\left\{ L_k'[u_c' f_l' - u_l'] + G_k'[u_g' - u_c'] - \frac{H_k'}{H_\tau'} \left(L_\tau'[u_c' f_l' - u_l'] + G_\tau'[u_g' - u_c'] \right) \right\} = 0.$$

This equation equates (weighted) *distortions* today and tomorrow. The distortions appear inside the brackets, and they are all zero in the unrestricted optimum. There are two kinds of distortions: those in the labor-leisure choice, and those in the private-public consumption choice. In the lump-sum tax case, the former were zero, and we ended up with a simple equation.

In the case of income taxes (which we could have derived above), the equation would be more complicated, because it would involve a distortion to the consumption-savings choice as well.

One can use the FOC for labor to simplify this expression somewhat:

$$L_\tau[u_c f_l \Psi] + G_\tau[u_g - u_c] + \beta H_\tau \left\{ (L'_k - \frac{H'_k}{H'_\tau} L'_\tau)[u'_c f'_l \Psi'] + (G'_k - \frac{H'_k}{H'_\tau} G'_\tau)\tau[u'_g - u'_c] \right\} = 0,$$

where Ψ and Ψ' denote the equilibrium tax rates in the current and next period, respectively.

This equation does not seem to offer more simplifications using first-order conditions: if the current-valued terms sum to zero for each k, the future-valued terms will not, unless $L_\tau/L_k = G_\tau/G_k$ (how much τ has to change in response to a change in k to keep labor constant is the same amount as it has to change to keep public expenditure constant).

Can we conclude from the above that the commitment solution to the labor taxation problem is not time-consistent? Not having analyzed the commitment solution it is hard to tell—we have to do it. Will the answer depend on whether or not there is government debt? The case with government debt is well-known not to be time-consistent (see Lucas and Stokey).

Finally, notice that the time-consistent labor taxation case is somewhat different from the income taxation case in one respect: the steady state interest rate is more easily obtained here (since the savings choice is undistorted). However, to find the level of capital and labor, one again needs to find the derivatives of the decision rules, which in turn will depend on the derivative of the Ψ function.