

1 Answer Keys for Homework 5

2 Exercise 1:

1. Notice that this is a particular case of that studied in Homework 4
2. Capital Accumulation Equation
 - Write down the budget constraint that the young face.
 - Write down the budget constraint that the old face.
 - Use the above to obtain the individual's lifetime budget constraint (by substituting for " a_{t+1} ").
 - State the maximization problem. It should read:
$$\max_{c_o, c_y} \{c_y^{0.5} c_o^{0.5}\}$$
 subject to: $w = c_y + \frac{c_o}{1+r_{t+1}}$
 - Solve for c_y from the lifetime budget constraint (so you have an expression in two variables that you are maximizing over) and replace it inside the utility function.
 - Take First Order Conditions and solve for individual demand functions (c_y, c_o) as a function of parameters. You should obtain the following: $c_y = 0.5w$, $c_o = 0.5(1+r)w$, $a_{t+1} = w - c_y = 0.5w$.
 - Find out w and r from the following maximization problem: $\max_{L, K} \{10K^{0.2}L^{0.8} - wL - rK\}$
 - Assume that the supply of labor is fixed at N (then $L = N$). After some algebra you can solve for the accumulation equation, $K_{t+1} = (0.5)(10)(0.8)K_t^{0.2}N^{0.8} = 4K_t^{0.2}N^{0.8}$. For more details, you can see chapter 3.
3. Steady State (Normalize $N = 1$). Set $K_{t+1} = K_t = \bar{K}$ and solve for the other variables replacing this into the production function, the expression for wages and interest and the consumption levels. You should get:
$$\bar{K} = 4^{\frac{1}{1-0.2}} = 5.6569$$
$$\bar{Y} = 10(5.6569)^{0.2} = 14.142$$
$$\bar{w} = 0.8(10)(5.6569)^{0.2} = 11.314$$
$$\bar{r} = 0.2(10)(5.6569)^{-0.8} = 0.5$$
$$\bar{c}^y = 0.5(11.314) = 5.657$$
$$\bar{c}^o = 0.5(1+0.5)(11.314) = 8.4855$$
4. Initial output is: $Y_o = 0.1\bar{Y} = 1.4142$. You need to find out the number of periods that will take to the economy to reach a level within 10% of the steady state output level: $0.9\bar{Y} = 12.7278$.
 - From the production function formula, you can solve for K_0 using the value of Y_o given above.

- Using the capital accumulation equation you can calculate values of capital in the first period, second period, and so on.
 - Replace these values in the production function to calculate the level of output in each period.
 - You will notice that after two periods output will be $Y = 12.898$.
5. Repeat the procedure with the new production function. In this case it will take 14 periods to reach a level within 10% of the steady state output level.

2.1 Exercise 2

- Write down the consolidated budget constraint and the maximization problem, as in exercise 1.

- Find out the capital accumulation equation solving the individual's maximization problem and knowing that firms maximize profits (you can assume that there is 1 person in the economy that supplies 1 unit of labor). For this specific problem, the capital accumulation equation is $K_{t+1} = B(1 - \beta)A_t K_t^\beta L^{1-\beta}$

- Notice that $\frac{A_{t+1}}{A_t} = 1 + a$. Hence, technology grows at a constant rate. From page 79 of the book, you know that if the technological progress is constant, the economy will eventually converge to a steady state. This steady state features growth at a constant rate in aggregate variables (as output and capital). At this point, you must realize that you are looking for a *constant rate of growth* of aggregate variables.

- Let g be the (unknown) growth rate of capital in steady state. Using the budget constraints and the capital accumulation equation you should be able to prove that capital and output grow at the rate a in the long run (i.e. $\frac{K_{t+1}}{K_t} = \frac{Y_{t+1}}{Y_t} = \frac{A_{t+1}}{A_t} = 1 + a$).

- In the long run, $K_t = \frac{K_0}{A_0} A_0$

2.2 Exercise 3

1. -Assume that there are N individuals in this economy before the immigration .

-Solve the individual maximization problem assuming that immigrants and initial citizens behave the same (i.e. they have the same utility function, etc.).

-Find an expression for the steady state level of capital per capita ($\bar{k} = \frac{\bar{K}}{N}$). Use the fact that immigrants arrive in period 1 and can work and save.

-Notice that \bar{k} is independent of the population size.

2. -Consider what happens to the capital/labor ratio in period 1.

-Determine the effect of this on the returns on capital in period 1 using a standard production function.

- Compare consumption of the initial old with and without immigration and realize that former is higher.
 - Conclude that Initial old will be better off (i.e. welfare increases).
3. -Consider what happens to the capital/labor ratio in period 1.
 - Determine the effect of this on the wages in period 1 using a standard production function.
 - Compare utility of the initial young with and without immigration and realize that former is lower.
 - Conclude that initial young will be worse off (i.e. welfare decreases).
 4. -Consider what happens to the capital/labor ratio in period $t > 1$. Use the fact that immigrants have as many children as initial citizens.
 - Determine the effect of this on relevant variables (wages, interest rates, capital accumulation) using a standard production and utility functions.
 - Compare utility of the a generation t individual with and without immigration and realize that they are equivalent.
 - Conclude that future generations will be unaffected by immigration.

2.3 Exercise 4

1. -Write down the budget constraint that the young face.
 - Write down the budget constraint that the old face.
 - Notice that if an individual saved a units when young, when old he or she will have $(1 - \delta + r_{t+1})a$ units.
2. -Calculate the individual's lifetime budget constraint (by substituting for " a_{t+1} ") and state the maximization problem. It should read:

$$\max_{c_o, c_y} \{c_y^\alpha c_o^{1-\alpha}\}$$
 subject to the budget constraint.
 - Solve for c_y from the lifetime budget, replace it inside the utility function and take first order conditions or set the $MRS =$ return on savings (which as you should notice is no longer $1 + r_{t+1}$)
 - Solve for a_{t+1} and find an expression for aggregate capital.
 - Find the steady state level of capital (setting $K_{t+1} = K_t = \bar{K}$).
 - Calculate the steady state levels of output, wages, interest rates and consumption.
 - Notice that the capital accumulation equation and the steady state levels are unaffected by the depreciation rate. The only exception is the steady state level of consumption by old individuals.
3. - Set $K_{t+1} = K_t = \bar{K}$.
 - Using the formulas provided in the question sheet find out the level of gross and net investment.

4. -Assume that individuals save one unit. In this model, after one period they will have $(1 - \delta)(1)$ units of capital.
- If one period in our model is equivalent to 30 years and depreciation is 10% per year, then by saving 1 unit today, individuals will have $(1 - 0.1)^{30}(1)$ units of capital after 30 years (or one period). Therefore, a realistic value of δ would be 0.9576 (be sure of understanding how to get this number).
5. -Let the production function be $Y = AK_t^\beta L_t^{1-\beta}$. Using the definition of capital share show that $\beta=0.3$.
- The annual interest rate net of depreciation is 4%. Hence, if an individual saves one unit of capital today, he or she will have $(1.04)^{30}$ units after thirty years. Using this and recalling that for every unit saved the return is $1 - \delta + r$, you should be able to find out that $r = 3.2$.
- Notice that $\beta \frac{\bar{Y}}{\bar{K}} = 1 - \delta + \bar{r}$. From here, you can find the capital-output ratio in steady state.
- Notice that if in one period output is \bar{Y} , and a period is equivalent to 30 years, then annual output is $\bar{Y}/30$. Show that the implied ratio of capital stock to annual output will be about 3.
6. Notice that aggregate consumption plus investment must equal total output.
- Solve for aggregate consumption as a function of aggregate investment and output.
- Calculate investment on steady state (don't forget that depreciation is non-zero).
- The ratio of total consumption to total output equals 0.9114
- Use the capital-output ratio in steady state