Answer Keys for Homework 1

2. Production Function

a) Cobb-Douglas production function $F(K,L) = K^{\theta} L^{1-\theta}$, $0 < \theta < 1$.

i. The marginal productivity of labor and capital are, respectively,

$$MP_K = \frac{\partial F(K,L)}{\partial K} = \theta K^{\theta-1} L^{1-\theta} = \theta \left(\frac{L}{K} \right)^{1-\theta},$$

$$MP_L = \frac{\partial F(K,L)}{\partial L} = (1-\theta) K^{\theta} L^{-\theta} = (1-\theta) \left(\frac{K}{L} \right)^{\theta}.$$

There are decreasing returns to capital and decreasing returns to labor.

$K \uparrow$, $MP_K \downarrow$,

$L \uparrow$, $MP_L \downarrow$.

ii. The marginal productivity of labor is increasing in capital and vice versa,

$K \uparrow$, $MP_L \uparrow$,

$L \uparrow$, $MP_K \uparrow$.

iii. The average productivity of labor and capital are, respectively,

$$AP_L = \frac{F(K,L)}{L} = \frac{K^{\theta} L^{1-\theta}}{L} = \left(\frac{K}{L} \right)^{\theta},$$

$$AP_K = \frac{K^{\theta} L^{1-\theta}}{K} = \left(\frac{L}{K} \right)^{1-\theta}.$$

The average productivity of labor is increasing in capital, and vice versa,

$K \uparrow$, $AP_L \uparrow$,

$L \uparrow$, $AP_K \uparrow$.

iv. There are constant returns to scale

$$F(xK, xL) = (xK)^{\theta}(xL)^{1-\theta} = x^{\theta} K^{\theta} x^{1-\theta} L^{1-\theta} = x^{\theta+(1-\theta)} K^{\theta} L^{1-\theta} = x\,F(K,L).$$

If labor and capital are changed by some factor x, then output is affected in the same proportion, x.

v. If the firm purchases inputs so as to set each input price equal to its marginal product, it will make zero profits,

$PM_L = (1 - \theta) K^{\theta} L^{-\theta}$ = w, Wage,
$PM_K = \theta K^{\theta-1} L^{1-\theta}$ = r, Rental price of capital,

$\Pi = F(K,L) - Lw - Kr$, Profits,
$\Pi = F(K,L) - L(1 - \theta) K^{\theta} L^{-\theta} - K\theta K^{\theta-1} L^{1-\theta}$,
$\Pi = F(K,L) - (1-\theta) K^{\theta} L^{1-\theta} - \theta K^{\theta} L^{1-\theta}$,
$\Pi = F(K,L) - F(K,L) = 0.$

vi. The capital and labor shares of income are, respectively,

$$\frac{Lw}{F(K,L)} = \frac{L(1-\theta)(K^{\theta}L^{-\theta})}{K^{\theta}L^{1-\theta}} = \frac{(1-\theta)(K^{\theta}L^{1-\theta})}{K^{\theta}L^{1-\theta}} = (1-\theta),$$
$$\frac{Kr}{F(K,L)} = \frac{K\theta(K^{\theta-1}L^{1-\theta})}{K^{\theta}L^{1-\theta}} = \frac{\theta(K^{\theta}L^{1-\theta})}{K^{\theta}L^{1-\theta}} = \theta.$$

The capital and labor shares of income do not depend on the inputs.

b) Production function F(K,L) = AK + BL.

i. The marginal productivity of labor and capital are, respectively,

MP_K = A,
MP_L = B,

Since MP_K is constant, there are no decreasing returns to capital,
Since MP_L is constant, there are no decreasing returns to labor.

ii. The marginal productivity of labor and capital are,

MP_K = A,
MP_L = B,

The marginal productivity of labor is not increasing in capital,
The marginal productivity of capital is not increasing in labor.

iii. The average productivity of labor and capital are, respectively,

$$AP_L = \frac{AK + BL}{L} = A\left(\frac{K}{L}\right) + B,$$
$$AP_K = \frac{AK + BL}{K} = A + B\left(\frac{L}{K}\right).$$

The average productivity of labor is increasing in capital, and vice versa,

$K \uparrow$, $AP_L \uparrow$,

$L \uparrow$, $AP_K \uparrow$.

iv. There are constant returns to scale

$F(xK, xL) = A\,(xK)+B\,(xL) = xAK+ xBL = x\,(AK + BL) = x\,F(K,L).$

v. If the firm purchases inputs so as to set each input price equal to its marginal product, it will make zero profits,

$PM_L = B$ = w, Wage,
$PM_K = A$ = r, Rental price of capital,

$\Pi = F(K,L) - wL - rK$, Profits,
$\Pi = F(K,L) - BL - AK$,
$\Pi = F(K,L) - (AK + BL)$,
$\Pi = F(K,L) - F(K,L) = 0.$

vi. The capital and labor shares of income are, respectively,

[wL / (AK + BL)] = [BL / (AK + BL)],

[rK / (AK + BL)] = [AK / (AK + BL)],

The capital and labor shares of income DEPEND on the inputs.