Lecture 4

Introduction to Mean-Variance Efficient Portfolios

AIM OF LECTURE 4

- Introduce the intuitive idea behind efficient portfolios
- Formulate the optimisation problem in order to find efficient portfolios
- Learn how to differentiate with respect to vectors

4.1 MEAN-VARIANCE EFFICIENT PORTFOLIOS

Throughout this course we will concentrate on situations where investors care only about mean and variance of returns. Remember from Lecture 3 that this is the case when either returns on assets have jointly normal distribution or when individuals have quadratic utility functions [Exercise 3.4]. From now on we implicitly rely on either of those assumptions, and go on to study mean-variance efficient portfolios.

Remember from Lecture 1 that a portfolio is a vector of weights:

$$\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \text{ with the restriction } \mathbf{1'w} = 1, \text{ i.e. } \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = 1$$

Also, remember that the <u>expected return</u> on a portfolio is

$$E[\tilde{R}_p] = \boldsymbol{w'R} = \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix} \begin{bmatrix} E[\tilde{R}_1] \\ E[\tilde{R}_2] \\ \vdots \\ E[\tilde{R}_n] \end{bmatrix}$$

and the <u>variance of return</u> on a portfolio is:

$$\sigma_p^2 = \boldsymbol{w'\Sigma w} = \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \dots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_2^2 & \dots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \dots & \sigma_n^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

In Lecture 1 we calculated mean and variance of different portfolios. By varying the weights potentially almost any pattern of return and risk can be obtained. We are most interested in mean-variance efficient portfolios.

4.1.1 Mean-Variance Inefficiency

We will now give an example of a portfolio that is not mean-variance efficient. Consider two portfolios, 1 and 2. The expected return and variance on the first is 10% and 50, respectively, while the expected return and variance on the second is 8% and 100, respectively. That is, $E[\tilde{R}_1]$=10%, σ_1^2=50, $E[\tilde{R}_2]$=8%, and σ_2^2=100. Then clearly the second portfolio is mean-variance inefficient. We can get both higher expected return, and lower risk, by investing in portfolio 1 rather than in portfolio 2.

4.1.2 Mean-Variance Efficiency

Definition A portfolio p (with mean $E[\tilde{R}_p]$ and variance σ_p^2) is mean-variance efficient if either

(i) one cannot create another portfolio q with greater expected return than p, and with no more variance than p (i.e. one cannot have $E[\tilde{R}_q] > E[\tilde{R}_p]$ and $\sigma_q^2 \leq \sigma_p^2$).

or

(ii) one cannot create another portfolio q with lower variance than p, and with no less expected return than p (i.e. one cannot have $\sigma_q^2 < \sigma_p^2$ and $E[\tilde{R}_q] \geq E[\tilde{R}_p]$).

4.1.3 Finding Mean-Variance Efficient Portfolios

The definition suggests how to find efficient portfolios. Take the second part, (ii). Fix any target level of expected return (say μ, which could be 5%, 10%, or anything) and try to arrange the portfolio weights so as to minimise the portfolio variance. This is a constrained minimisation problem: the objective function is $\boldsymbol{w}'\Sigma\boldsymbol{w}$, one constraint is $E[\tilde{R}_p]\geq\mu$, and the other constraint is that the weights sum to one, i.e. $\boldsymbol{w}'\mathbf{1}=1$. Remember that $E[\tilde{R}_p]\geq\mu$ can be written as $\boldsymbol{w}'\boldsymbol{R}\geq\mu$, where R is the vector of expected returns, $\left[E[\tilde{R}_1], E[\tilde{R}_2],..., E[\tilde{R}_n]\right]'$.
We write this problem as follows:

$$\min_{w} \; \boldsymbol{w}'\boldsymbol{\Sigma}\,\boldsymbol{w}$$

$$\text{s.t.} \;\; \boldsymbol{w}'\boldsymbol{R}\geq\mu \text{ and } \boldsymbol{w}'\mathbf{1}=1.$$

To solve the problem we write the Lagrangian

$$\mathfrak{L} = \boldsymbol{w}'\boldsymbol{\Sigma}\,\boldsymbol{w} + \lambda\{\mu - \boldsymbol{w}'\boldsymbol{R}\} + \gamma\{1 - \boldsymbol{w}'\mathbf{1}\}$$

and differentiate!

We must first be able to differentiate $\boldsymbol{w}'\Sigma\boldsymbol{w}$, $\boldsymbol{w}'\boldsymbol{R}$,and $\boldsymbol{w}'\mathbf{1}$ with respect to $\boldsymbol{w}$. This is the topic of the next section!

4.2 VECTOR DIFFERENTIATION

4.2.1 Differentiation of a Quadratic Form

We are after the gradient (denoted by ∇) which is

$$\nabla(\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}) = \begin{bmatrix} \dfrac{\partial(\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w})}{\partial w_1} \\ \dfrac{\partial(\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w})}{\partial w_2} \\ \vdots \\ \dfrac{\partial(\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w})}{\partial w_n} \end{bmatrix}$$

The gradient is just the vector of derivatives of $\boldsymbol{w}'\Sigma\boldsymbol{w}$ with respect to each of the elements of $\boldsymbol{w}$, i.e. with respect to w_1, w_2, etc.

The rule of differentiation tells us that:

$$\begin{aligned} \nabla(\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}) &= \boldsymbol{\Sigma}\boldsymbol{w} + \boldsymbol{\Sigma}^{\top}\boldsymbol{w} \\ &= \left(\boldsymbol{\Sigma} + \boldsymbol{\Sigma}^{\top}\right)\boldsymbol{w} \end{aligned}$$

Note that, since Σ is a variance-covariance matrix it is symmetric and therefore $\Sigma = \Sigma^{\top}$ and we have $\nabla(\boldsymbol{w}'\Sigma\boldsymbol{w}) = 2\Sigma\boldsymbol{w}$.

The derivation of the rule of differentiation is in the appendix to this lecture.

4.2.2 Differentiation of an Inner Product (Dot Product)

Differentiating an inner product, like $\boldsymbol{w}'\boldsymbol{R}$, is easy:

$$\nabla(\boldsymbol{w}'\boldsymbol{R}) = \boldsymbol{R}$$

This can be seen as follows. We have $\boldsymbol{w}'\boldsymbol{R} = w_1R_1+w_2R_2+\ldots+w_jR_j+\ldots+w_nR_n$.

(In our application the elements of $\boldsymbol{R}$ are the expected returns, so R_1 is $E[\tilde{R}_1]$, etc.)

$$\begin{array}{ll} \text{Then,} & \partial(\boldsymbol{w}'\boldsymbol{R})/\partial w_1 = R_1, \\ \text{and} & \partial(\boldsymbol{w}'\boldsymbol{R})/\partial w_2 = R_2, \\ \vdots & \quad\vdots \qquad\quad \vdots \quad \vdots \\ \text{and} & \partial(\boldsymbol{w}'\boldsymbol{R})/\partial w_j = R_j \\ \vdots & \quad\vdots \qquad\quad \vdots \quad \vdots \\ \text{and} & \partial(\boldsymbol{w}'\boldsymbol{R})/\partial w_j = R_j. \end{array}$$

These derivatives can be written in vector form:

$$\nabla(\boldsymbol{w}'\boldsymbol{R}) = \begin{bmatrix} \dfrac{\partial(\boldsymbol{w}'\boldsymbol{R})}{\partial w_1} \\ \dfrac{\partial(\boldsymbol{w}'\boldsymbol{R})}{\partial w_2} \\ \vdots \\ \dfrac{\partial(\boldsymbol{w}'\boldsymbol{R})}{\partial w_n} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1 \\ \boldsymbol{R}_2 \\ \vdots \\ \boldsymbol{R}_n \end{bmatrix} = \boldsymbol{R}$$

4.3 STUDY SUGGESTIONS

Make sure you remember the rules of differentiation for vector products (the one for a quadratic form, and the one for an inner product), and that you understand why the rules look like they do (by going through, step-by-step, the appendix to this lecture).
Make an attempt to solve the exercises that follow below!

4.4 NEXT TIME

Next time we will solve the minimum variance problem (applying the rules of differentiation), and analyse the resulting mean-variance efficient portfolios.

EXERCISES

Exercise 4.1 Find the derivatives with respect to $\boldsymbol{x}$ of the following:
(i) $\boldsymbol{x}'\boldsymbol{x}$
(ii) $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x}$
where $\boldsymbol{x}$ is a nx1 vector, and $\boldsymbol{A}$ and $\boldsymbol{B}$ are nxn matrices.

Exercise 4.2 What are the similarities/differences between the differentiation in Exercise 4.1, and the differentiation with respect to a scalar x of the following:
(i) x^2
(ii) ax^2bx^2
where a and b are constants (scalars)?

Exercise 4.3 What are the implicit assumptions when formulating the portfolio-variance minimisation problem? What are the information requirements?

Exercise 4.4 Formulate the minimum-variance problem when short-selling is not allowed.

APPENDIX

Proof of $\nabla(\boldsymbol{w}'\Sigma\boldsymbol{w}) = (\Sigma+\Sigma^{\top})\boldsymbol{w}$

First we can postmultiply with the last vector

$$\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}\begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_n^2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}\begin{bmatrix} \sigma_1^2 w_1 + \sigma_{1,2}w_2 + \cdots + \sigma_{1,n}w_n \\ \sigma_{2,1}w_1 + \sigma_2^2 w_2 + \cdots + \sigma_{2,n}w_n \\ \vdots \\ \sigma_{n,1}w_1 + \sigma_{n,2}w_2 + \cdots + \sigma_n^2 w_n \end{bmatrix}$$

and then premultiply with the first vector

$$\begin{aligned} \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} = {} & w_1(\sigma_1^2 w_1 + \sigma_{1,2}w_2 + \ldots + \sigma_{1,j}w_j + \ldots + \sigma_{1,n}w_n) \\ & + w_2(\sigma_{2,1}w_1 + \sigma_2^2 w_2 + \ldots + \sigma_{2,j}w_j + \ldots + \sigma_{2,n}w_n) \\ & \vdots \\ & + w_j(\sigma_{j,1}w_1 + \sigma_{j,2}w_2 + \ldots + \sigma_j^2 w_j + \ldots + \sigma_{j,n}w_n) \\ & \vdots \\ & + w_n(\sigma_{n,1}w_1 + \sigma_{n,2}w_2 + \ldots + \sigma_{n,j}w_j + \ldots + \sigma_n^2 w_n) \end{aligned}$$

We see that a portfolio weight enters twice. For example, w_j enters multiplied by $\sigma_{j,1}w_1+\sigma_{j,2}w_2+\ldots+\sigma_j^2 w_j+\ldots+\sigma_{j,n}w_n$, and it also appears in each line (i.e. in the first line it is multiplied by $\sigma_{j,1}$ and by w_1, in the second line it is multiplied by $\sigma_{j,2}$ and by w_2, and so on).

Taking the derivative with respect to w_j gives

$$\frac{\partial \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}}{\partial w_j} = (\sigma_{j,1}w_1 + \sigma_{j,2}w_2 + \ldots + \sigma_j^2 w_j + \ldots + \sigma_{j,n}w_n) + (w_1\sigma_{1,j} + w_2\sigma_{2,j} + \ldots + w_j\sigma_j^2 + \ldots + w_n\sigma_{n,j})$$

Notice that this can be written as the j^{th} row of Σ times $\boldsymbol{w}$ plus $\boldsymbol{w}'$ times the j^{th} column of Σ.

Taking derivatives with respect to w_1, w_2,...,w_n in turn gives:

$\partial(\boldsymbol{w}'\Sigma\boldsymbol{w})/\partial w_1$ = 1^{st} row of Σ times $\boldsymbol{w}$ + $\boldsymbol{w}'$ times 1^{st} column of Σ.
$\partial(\boldsymbol{w}'\Sigma\boldsymbol{w})/\partial w_2$ = 2^{nd} row of Σ times $\boldsymbol{w}$ + $\boldsymbol{w}'$ times 2^{nd} column of Σ.
$\vdots$ $\quad\vdots\quad$ $\vdots$ $\quad\vdots\quad$ $\vdots$
$\partial(\boldsymbol{w}'\Sigma\boldsymbol{w})/\partial w_n$ = n^{th} row of Σ times $\boldsymbol{w}$ + $\boldsymbol{w}'$ times n^{th} column of Σ.

Notice that the first term of each row on the right hand side can be obtained by postmultiplying Σ with $\boldsymbol{w}$ (then each of the rows of Σ will be multiplied by $\boldsymbol{w}$). The result is a column vector. The second term of each of the rows on the right hand side can be obtained by premultiply Σ by $\boldsymbol{w}'$ (then each of the columns of Σ will be multiplied by $\boldsymbol{w}$). However the result is a row vector. To write it as a column vector we need to transpose it.

Then we can write

$$\begin{bmatrix} \dfrac{\partial(\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w})}{\partial w_1} \\ \dfrac{\partial(\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w})}{\partial w_2} \\ \vdots \\ \dfrac{\partial(\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w})}{\partial w_n} \end{bmatrix} = \boldsymbol{\Sigma}\boldsymbol{w} + \left(\boldsymbol{w}'\boldsymbol{\Sigma}\right)^{\top}$$

The left hand side is the gradient $\nabla(\boldsymbol{w}'\Sigma\boldsymbol{w})$. The transpose of $\boldsymbol{w}'\Sigma$ is $\Sigma^{\top}\boldsymbol{w}$. Then we have $\nabla(\boldsymbol{w}'\Sigma\boldsymbol{w}) = \Sigma\boldsymbol{w}+\Sigma^{\top}\boldsymbol{w} = (\Sigma+\Sigma^{\top})\boldsymbol{w}$. QED