Lecture 3

Choice Under Uncertainty: Risk and Insurance Premia

AIM OF LECTURE 3

- Introduce the concept of risk aversion
- Provide measures of risk aversion
- Introducing the risk premium
- Solve for the Markowitz risk premium and analyze its components
- Introduce Mean-Variance analysis as a choice criterion

3.1 RISK AVERSION

(Section C in Chapter 4, C&W)

3.1.1 Definitions

Definition 1 An individual is

risk averse if he is *not willing to accept* any actuarially fair gamble.
risk neutral if he is *indifferent* to any actuarially fair gamble.
risk lover if he is *willing to accept* any actuarially fair gamble.

Definition 2

$U(E[\tilde{W}]) > E[U(\tilde{W})]$ Risk aversion
$U(E[\tilde{W}]) = E[U(\tilde{W})]$ Risk neutrality
$U(E[\tilde{W}]) < E[U(\tilde{W})]$ Risk loving

Actuarially fair gamble

Two payoffs $R_1<0$, $R_2>0$ (i.e. one is negative). The gamble is receiving R_1 (actually paying $-R_1$) with probability α and receiving R_2 with probability $1-\alpha$, and the expected return of the gamble is zero

$$E[\tilde{R}] = \alpha R_1 + (1-\alpha)R_2 = 0$$

Exercise 3.1 Show that Definition 1 implies Definition 2 and vice versa.

Exercise 3.2 Show that Definitions 1 and 2 imply a concave utility function if the individual is risk averse, a linear utility function if the individual is risk neutral, and a convex utility function if the individual is risk lover. Hint: Jensen's inequality.

We shall illustrate Exercise 3.2 graphically. Let $W_1 = W_0(1 + R_1)$ and $W_2 = W_0(1 + R_2)$, where (R_1, R_2) are the payoffs of an actuarially fair gamble. Then clearly $E[\tilde{W}] = W_0$ and $W_1 < E[\tilde{W}] < W_2$. We plot a utility function in a diagram. The vertical axis shows the value of the utility function and the horizontal axis the value of wealth under different outcomes. Since expected utility of the gamble is a *linear combination* of the utilities obtained in state 1 and state 2, i.e.

$$E[U(\tilde{W})] = \alpha U(W_1) + (1\text{-}\alpha)U(W_2)$$

expected utility must lie on a straight line connecting $U(W_1)$ and $U(W_2)$, denoted by the dotted line in the figure below.

Figure 3.1

U

$U(\tilde{W})$

$U(W_2)$

$U(W_0)$

$E[U]$

$U(W_1)$

W_1 W_0 W_2 $\tilde{W}$

3.1.2 Absolute and Relative Risk Aversion (ARA, RRA) and Risk Tolerance (RT)

$$ARA \equiv -\frac{U''(W)}{U'(W)}, \quad RRA \equiv -\frac{U''(W)}{U'(W)}W, \text{ and } RT \equiv \frac{1}{ARA}$$

are measures of the <u>curvature</u> of the utility function. We will examine these measures more closely when we look at optimal portfolio choice. Absolute risk aversion appears in the expression for the risk premium for "small" risk, see below.

Exercise 3.3 Show that just the second derivative, U'', alone is *not enough* for a definition risk aversion. Show why we need to divide by U'
Hint: To what extent is U unique?

3.2 RISK AVERSION IN THE SMALL AND IN THE LARGE

(Section D in Chapter 4, C&W)

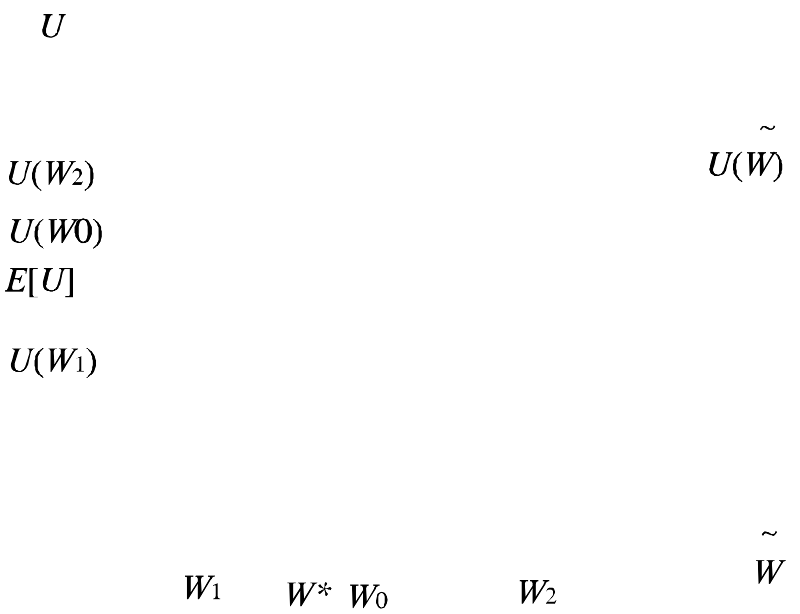
How can we define the *risk premium* and what can we say about it under the most general conditions? What we mean by the "most general conditions" is that we don't make the (rather restrictive) assumptions underlying e.g. the CAPM.

3.2.1 Certainty equivalent

Consider two investment possibilities. One is a risky investment opportunity, the other is certain. The risky investment pays off W, the certain pays off W^*. What is the level of the payoff of the certain investment, i.e. the level of W^*, that would make the individual *indifferent* between the two investment possibilities? We call this level of payoff the *certainty equivalent*.

> **Definition** The *certainty equivalent* is the amount of a certain payoff that makes an individual *indifferent* between undertaking the risky investment and taking the certain payoff, i.e.
>
> $$U(W^*) = E[U(\tilde{W})].$$

Graphically:



Mathematically: Suppose $\tilde{W} = \begin{cases} W_1, \ prob = \alpha \\ W_2, \ prob = 1-\alpha \end{cases}$, then

$$U(W^*) = \alpha U(W_1) + (1\text{-}\alpha)U(W_2) < U(E[\tilde{W}]) \Leftrightarrow W^* < E[\tilde{W}].$$

3.2.2 Risk and Insurance premium

> **Definition** The (Markowitz) *risk premium* is the amount an individual is willing to *pay* to avoid a risky situation.

Risky "situation" is $\tilde{W}$ (the ensured situation is W_0 - π)

$\nearrow$ initial wealth $\nwarrow$ insurance premium

Examples: Investment in shares
(amount invested in shares = W_0, cost of buying a put option = π)
Home insurance
(value of the property = W_0, cost of buying an insurance contract = π)

Deriving the risk premium for small risk

Let π (the risk premium) be such that the individual is *indifferent* between the risky situation and the ensured one and assume that $E[\tilde{W}]=W_0$, then clearly π satisfies

$$E[U(W_0 - \pi)] = E[U(\tilde{W})].$$

$\nwarrow \nearrow$ these are certain!

N.B. $W_0 - \pi$ equals the *certainty equivalent* W^*.

First-order Taylor approximation of left-hand side

$$U(W^*) \approx U(W_0) + U'(W_0)(W^*-W_0) = U(W_0) - U'(W_0)\pi$$

Second-order Taylor approximation of right-hand side (ignoring $E[\ ]$ for the moment)

$$U(\tilde{W}) \approx U(W_0) + U'(W_0)(\tilde{W}-W_0) + \tfrac{1}{2}U''(W_0)(\tilde{W}-W_0)^2$$

taking expectations[1]

$$E[U(\tilde{W})] \approx U(W_0) + U'(W_0)\underbrace{E[\tilde{W}-W_0]}_{=0} + \tfrac{1}{2}U''(W_0)\underbrace{E[(\tilde{W}-W_0)^2]}_{=\sigma_W^2}$$

$$\Rightarrow \qquad E[U(\tilde{W})] \approx U(W_0) + \tfrac{1}{2}U''(W_0)\sigma_W^2$$

Combining left- and right-hand sides gives

$$\pi \approx -\frac{\sigma_W^2}{2}\frac{U''(W_0)}{U'(W_0)} = \frac{\sigma_W^2}{2}\cdot ARA$$

[1] $U(W_0)$, $U'(W_0)$, $U''(W_0)$ are *constants* since W_0 is not stochastic.

The risk premium is *proportional* to ARA. If ARA is large the individual pays more to avoid the risky situation. If σ_W^2 is large (risk is larger) $\Rightarrow$ the risk premium larger.

NB. This is risk aversion "in the small," because of the Taylor approximation is valid only for "small" deviations from W_0. Read section D of Chapter 4 in C&W to see how the approximation becomes worse if risk is "large."

3.3 MEAN-VARIANCE AS A CHOICE CRITERION

(Section F in Chapter 4, C&W)

Idea: individuals prefer more to less, but have aversion against risk (variation in return). We often represent the risk by the *variance* of return. Is there a way in which we can make the variance the only relevant measure of risk, so that we could represent preferences such that individuals make choices by looking at mean and variance only? There are two ways: either we assume that random returns are *jointly normally distributed* or we assume that individuals have *quadratic utility*, (or we assume both). The textbook shows how one may proceed if returns are jointly normal. However, it's quite easy to do with quadratic utility. We'll do it later on, for time being do make an attempt yourself.

Exercise 3.4 Show that preferences may be represented as mean-variance preferences if the utility function is quadratic: $U(W) = aW - \frac{b}{2}W^2$. Hint: Proceed from equation (4.14) in Copeland and Weston.

3.4 SOME UTILITY FUNCTIONS IN FINANCE

Quadratic $\quad U(W) = aW - \frac{b}{2}W^2$

Negative exponential $\quad U(W) = -\frac{e^{-\lambda W}}{\lambda} \quad \Leftrightarrow$ CARA

Iso elastic (power function) $\quad U(W) = \frac{W^{1-\gamma}}{1-\gamma} \quad \Leftrightarrow$ CRRA

Hyperbolic Absolute Risk Aversion (HARA)

$$U(W) = \frac{\gamma}{1-\gamma}\left(a + \frac{b}{\gamma}W\right)^{1-\gamma}$$

Quadratic, CARA, CRRA are *special cases* of HARA.

Solve the following exercise only if you have spare time:

Exercise 3.5
(*a*) What is ARA for HARA?
(*b*) What can be said about Risk Tolerance (RT) for HARA?
(*c*) Show that negative exponential utility *implies* CARA and that iso-elastic utility *implies* CRRA. (The implications go the other way as well, but showing that is beyond the scope of this course).

3.5 SOME PARADOXES

Read Sections G and H in Chapter 4, Copeland & Weston, on your own.

3.6 STUDY SUGGESTIONS

Read through the relevant chapters/parts as indicated in these notes. Make an attempt to solve the exercises above. Make sure that you understand the concepts: risk aversion, risk premium, and the certainty equivalent, and that you can explain these concepts diagrammatically. Go through, step by step, the derivation of the Markowitz risk premium. Do the exercises below (important!).

3.7 NEXT TIME

Next time we are going to analyse the properties of portfolios that are mea-variance efficient.

EXERCISES

The following exercises in Copeland and Weston are suggested:

4.1, 4.2, 4.3, 4.5, 4.9.