Lecture 2

Return, Risk and Choice Under Uncertainty

AIM OF LECTURE 2

- Calculate return and risk of portfolios using matrix algebra
- Revise portfolio formation in the context of short sales
- Revise expected utility and provide a numerical example
- Show the difference between expected utility and utility of expected value in the context of a numerical example

2.1 RETURN AND RISK USING MATRIX ALGEBRA

A very convenient way to calculate expected return and variance of portfolios is to use matrix algebra. In order to do so we need the vector of expected returns and the variance-covariance matrix.

Vector of expected return, n assets:

$$\boldsymbol{R} = \begin{bmatrix} E[\tilde{R}_1] \\ E[\tilde{R}_2] \\ \vdots \\ E[\tilde{R}_n] \end{bmatrix}$$

Variance-covariance matrix:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_n^2 \end{bmatrix}$$

A portfolio is a vector of weights:

$$\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$, with the restriction $\mathbf{1'w} = 1$, i.e. $$\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = 1$$

The expected return on a portfolio is:

$$E[\tilde{R}_p] = \boldsymbol{w}'\boldsymbol{R} = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} E[\tilde{R}_1] \\ E[\tilde{R}_2] \\ \vdots \\ E[\tilde{R}_n] \end{bmatrix}$$

The variance of return on a portfolio is:

$$\sigma_p^2 = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_n^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

In our example above we have:

$$E[\tilde{R}_p] = \boldsymbol{w}'\boldsymbol{R} = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 12 \\ 2.5 \end{bmatrix} = 4.4$$

$$\sigma_p^2 = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 276 & 105 \\ 105 & 41.25 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} = 71.04$$

Exercise 2.1 Given the variance-covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 24 & -10 & 25 \\ -10 & 75 & 32 \\ 25 & 32 & 12 \end{bmatrix}$$

a) Calculate the variance of an equally weighted portfolio.
b) Calculate the covariance of a portfolio that has 10% in asset 1, 80% in asset 2, and 10% in asset 3 with a second portfolio that has 125% in asset 1, -10% in asset 2, and -15% in asset 3.

2.2 SHORT SELLING

In the last lecture we calculated return and risk on portfolios when an individual was investing positive amounts in the various assets. In those cases each portfolio weight is positive, i.e. $w_j \geq 0$ for all assets j. In exercise 2.1 you worked with a case where some weights were negative, and some were greater than unity. This is a case when an individual has done short sales, i.e. sold an asset (s)he does not own, and will buy the asset later on to deliver it to the buyer.

We will do two examples, based on the assets analysed in the last lecture, to illustrate this. We will use the example involving BA shares and the fixed-income security.

We need the following from last lecture:

The distribution of the value of BA shares

$$P\{\tilde{V}_{BA} = £5.4\} = 0.3$$
$$P\{\tilde{V}_{BA} = £6.6\} = 0.3$$
$$P\{\tilde{V}_{BA} = £7.8\} = 0.4$$

The interest rate on the fixed income security

$$R_f = 6\%$$

The individual's initial endowment

$$W = £\ 8000$$

The initial share-price of BA shares

$$V_{BA} = £6.00$$

The expected return on BA shares (which we can work out from the above)

$$E[\tilde{R}_{BA}] = 12\%$$

Example (a)

The simplest example of short sales is when an individual borrows money to invest in risky assets.

Let the individual borrow £4000 at interest $R_f = 6\%$, and invest those £4000 plus initial endowment (£8000) in BA shares.

The individual then invests £4000+£8000 = £12000 in BA shares and thus buys 2000 BA shares (2000·£6=£12000). In state 1 those shares are valued 10800 (2000·£5.4=£10800), in state 2 they are valued 13200 (2000·£6.6=£13200), and in state 3 their value is 15600 (2000·£7.8=£15600). Regardless the state of nature, the individual has to pay back the loan with interest, i.e. (s)he has to pay back £4000 + £4000·6% = £4240.

We then have the following payoff table:

State:	BA-share	Loan	Portfolio
s_1 = low	£ 10,800	- £ 4,240	£ 6,560
s_1 = med.	£ 13,200	- £ 4,240	£ 8,960
s_3 = high	£ 15,600	- £ 4,240	£ 11,360

We can find the expected value and the expected return on this portfolio in the usual way:

Expected value is $E[\tilde{W}_p]$=0.3×6560+0.3×8960+0.4×11360= **£ 9200**

Expected return is $E[\tilde{R}_p]=(E[\tilde{W}_p]-W)/W$ = (9200-8000)/8000 = **15%**

We could have obtained the above expected portfolio return in the following way:

The weight invested in BA shares is w_{BA}=12000/8000 = 1.5, and the weight invested in the risk-free asset is w_f= -4000/8000 = -0.5.

Since we know the vector of expected returns, **R**=[12, 6]′, we can use the vector of portfolio weights,**w**=[1.5, -0.5]′, to obtain the expected return on the portfolio:

$E[\tilde{R}_p]=\mathbf{w'R}$=[1.5, -0.5][12, 6]′ = 1.5×12% + (-0.5)×6%=**15%**

Example (b)

Another example of short sales is when the individual in the beginning of the investment period sells BA shares, and invests in the fixed-income security. Since the individual does not own BA share, (s)he has to buy them at a later date to be able to deliver the shares to the person who bought them. Suppose the individual shortsells 1000 BA shares. The timing of events is as follows.

time

sell 1000 BA shares	return on fixed-income security = £ 840 ⇒ cash = £14840
invest £ 6000 (from BA shares) +£ 8000 (initial endowment) £14000 in fixed-income security	deliver 1000 BA shares, but has to purchase them first!

The individual shortsells 1000 BA shares, and receives £6000 as payment (1000·£6= £6000). The individual invests those £6000 plus initial endowment (£8000) in the fixed-income security, which earns interest R_f= 6%. In the next period (s)he receives the value of the fixed-income security £14840 (regardless the state of nature), and purchases BA shares (to deliver them to the initial buyer) at the going price. (i.e.invest those £4000 plus initial endowment (£8000) in BA shares. In state 1 it will cost the individual £5400 to purchase the shares (1000·£5.4=£5400), in state 2 it will cost £6600 (1000·£6.6=£6600), and in state 3 it costs £7800 (1000·£7.8=£7800).

We then have the following payoff table:

State:	BA-share	Fixed-income security	Portfolio
s_1 = low	-£ 5,400	£ 14,840	£ 9,440
s_1 = med.	-£ 6,600	£ 14,840	£ 8,240
s_3 = high	-£ 7,800	£ 14,840	£ 7,040

We can find the expected value and the expected return on this portfolio in the usual way:

Expected value is $E[\tilde{W}_p]$=0.3×9440+0.3×8240+0.4×7040= **£ 8120**

Expected return is $E[\tilde{R}_p]=(E[\tilde{W}_p]-W)/W$ = (8120-8000)/8000 = **1.5%**

We could have obtained the above expected portfolio return in the following way:

The weight invested in BA shares is w_{BA}=-6000/8000 = -0.75, and the weight invested in the risk-free asset is w_f= 14000/8000 = 1.75.

We use the vector of expected returns, **R**=[12, 6]′, together with the vector of portfolio weights,**w**=[-0.75, 1.75]′, to obtain the expected return on the portfolio:

$E[\tilde{R}_p]=\mathbf{w}'\mathbf{R}$=[-0.75, 1.75][12, 6]′ = -0.75×12% + 1.75×6%=**1.5%**

2.3 REVISION OF EXPECTED UTILITY

(Sections A and B in Chapter 4, C&W)

2.3.1 Definition of Expected Utility

Revise the axioms in C&W, though they are not central to this course.

Important is that you remember the definition of expected utility:

$$E[U(\tilde{W})] = \sum_{s=1}^{S} p_s U(W_s)$$

where W_s is wealth if state s occurs, and 1,..,S are the states of the world, and state s occurs with probability p_s. In our example above we had three states of the world, with p_1=0.3, p_2=0.3, and p_3=0.4.

Exercise 2.2 Assume an investor who has an initial wealth of 100. All of this is to be invested in either BT shares or BA shares.

a) Which investment does the investor prefer if she has a von Neumann-Morgenstern utility function of the form

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma}$$

with

i) γ = 0.5 ?

ii) γ = 1.5 ?

Give an interpretation of γ.

b) Can you find a portfolio of BT shares and BA shares which is preferred by the individual with γ = 0.5 to holding solely the share you found in question a)? [Hint: assume that shortsales is allowed so a portfolio weight can be negative (the weights only have to *sum* to one)].

c) Can you find a portfolio of BT shares and BA shares which is preferred by the individual with γ = 1.5 to holding solely the share you found in question a)? [Hint: assume that shortsales is allowed so a portfolio weight can be negative (the weights only have to *sum* to one)].

2.3.2 Numerical Example of Expected Utility

Suppose an individual has initial wealth of *W*, and invests in BA shares only, which is this person's expected utility of wealth if her von Neumann-Morgenstern utility function of the form

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma}$$

where γ = 0.5 ?

Remember, from the definition of expected utility, that we have to work out the utility in each state of nature, premultiply with respective probability, and sum over all states.

Wealth in state 1 is $W\cdot[1+\tilde{R}_{BA}|s_1] = W\cdot[1\text{-}10\%] = W\cdot[1\text{-}0.1] = W\cdot 0.9$.
Wealth in state 2 is $W\cdot[1+\tilde{R}_{BA}|s_2] = W\cdot[1+10\%] = W\cdot[1+0.1] = W\cdot 1.1$.
Wealth in state 3 is $W\cdot[1+\tilde{R}_{BA}|s_3] = W\cdot[1+30\%] = W\cdot[1+0.3] = W\cdot 1.3$.

Utility in state 1 is $U(W\cdot[1+\tilde{R}_{BA}|s_1]) = U(W\cdot 0.9) = [W\cdot 0.9]^{1-\gamma}/[1-\gamma]$
$= [W\cdot 0.9]^{0.5}/0.5$
$= 2\cdot W^{0.5}\cdot 0.9^{0.5}$
$= 1.8974\cdot W^{0.5}$
Utility in state 2 is $U(W\cdot[1+\tilde{R}_{BA}|s_2]) = U(W\cdot 1.1) = 2\cdot W^{0.5}\cdot 1.1^{0.5}$
$= 2.0976\cdot W^{0.5}$
Utility in state 3 is $U(W\cdot[1+\tilde{R}_{BA}|s_2]) = U(W\cdot 1.3) = 2\cdot W^{0.5}\cdot 1.3^{0.5}$
$= 2.2804\cdot W^{0.5}$

We then have the following table:

State:	BA-share	Wealth	Utility
s_1 = low	- 10%	$0.9\cdot W$	$1.89737\cdot W^{0.5}$
s_1 = med.	+ 10%	$1.1\cdot W$	$2.09762\cdot W^{0.5}$
s_3 = high	+ 30%	$1.3\cdot W$	$2.28035\cdot W^{0.5}$

We can find the expected utility of wealth (when investing in BA shares) in the usual way:

Expected utility is:

$E[U(\tilde{W})] = E[U(W\cdot[1+\tilde{R}_{BA}])]$
$= 0.3\cdot 1.8937\cdot W^{0.5} + 0.3\cdot 2.09762\cdot W^{0.5} + 0.4\cdot 2.28035\cdot W^{0.5}$
$\approx 2.11064\cdot W^{0.5}$

2.3.3 Utility of Expected value

When we look at utility of expected wealth, we look at a situation without uncertainty (in all states of nature the individual consumes a certain quantity of wealth). This must not be confused with expected utility, which evaluates uncertain wealth levels. We will analyse the exact difference in the next section, but for time being we can illustrate the point by using the numerical example above.

Utility of expected wealth is:

$U(E[\tilde{W}]) = U(E[W\cdot[1+\tilde{R}_{BA}]]) = U(W\cdot[1+E[\tilde{R}_{BA}]])$
$= U(W\cdot[1+0.12]) = U(W\cdot 1.12) = (1.12\cdot W)^{0.5}/0.5 = 2\cdot 1.12^{0.5}\cdot W^{0.5}$
$\approx 2.11660\cdot W^{0.5}$

Since $2.11660\cdot W^{0.5} > 2.11064\cdot W^{0.5}$, the individual prefers to get the expected return on BA shares for sure, rather than being under the risky situation when the return on BA shares change according to state of nature. This is because the individual is risk averse, and this is the topic of the next lecture!

2.4 STUDY SUGGESTIONS

Read through the relevant chapters/parts as indicated in these notes. Make an attempt to solve the exercises above. Make sure that you understand how shortselling can be done (and therefore why portfolio weights can be negative as well as greater than 100%). Make sure you understand how to calculate expected utility, and that the expected utility is different from the utility of expected wealth. Do the additional exercises below (important!).

2.5 NEXT TIME

Next time we are going to introduce the concepts certainty equivalent, Markowitz risk premium, and mean-variance analysis as a choice criterion. Revise Taylor approximation in Appendix D.

EXERCISES

Do exercises 2.1-2.2 in this lecture handout!