Lecture 13

Introduction to Continuous-Time Finance and Option Pricing

AIM OF LECTURE 13

- Become familiar with some continuous-time finance
- Learn how to form hedge portfolios with options
- Gain understanding of how a no-arbitrage argument underlies the Black-Scholes option-pricing formula
- Practice the Black-Scholes option pricing formula

13.1 INTRODUCTION TO CONTINUOUS-TIME FINANCE

On financial markets assets are traded at very frequent intervals of time (fractions of minutes). A reasonable approximation (theoretically) is to let the interval of time go arbitrarily close to zero, we are then in a world of continuous-time models. Technically these models are easier to handle, and the Black-Scholes formula is based on a continuous-time model.

The Risk-Free Rate of Interest

One of the first things that changes in continuous time is the rate of interest and discounting. First, let's take an example from the discrete-time case. Suppose you invest V_0 today and get the payoff in t years. The annual interest rate is R. You then have, at the end of t years

$$V_t = V_0(1+R)^t$$

However, with continuous compounding, we have instead

$$V_t = V_0\, e^{rt}$$

where r is the continuous-time interest rate. Combining the above relations we have

$$V_t/V_0 = (1+R)^{t} = e^{rt}$$

and taking logarithms: $t\ln(1+R) = rt$, or

$$\mathbf{ln(1+\mathit{R}) = \mathit{r}}$$

We can obtain the continuous-time interest rate by taking the natural logarithm of one plus the discrete-time interest rate. This is important because we only obtain market information about the discrete-time interest rate, while in the Black-Scholes analysis we need the continuous-time version.

Share Prices
The efficient market hypothesis suggests that share prices should follow random walks (with drift). The continuous-time analogue of a random walk is a *Brownian motion* (or Wiener process or Ito process). The process we will work with is a *Geometric Brownian Motion,* and it looks as follows:

$$ds = \mu s dt + \sigma s dz$$

$$dz = \varepsilon \sqrt{dt}$$

where ε is a standard-normal random variable, i.e. ε is normally distributed with mean zero ($E[\varepsilon]=0$), and unity variance ($VAR(\varepsilon)=1$).

The picture below depicts a Geometric Brownian Motion.

Figure 13.1

It looks strikingly like a financial time series!

Below is the FTSE 100-share index, daily observations between 19 Dec 1994 and 2 May 2001.

Figure 13.2

Properties implied by the Brownian Motion:
- continuous changes in asset prices;
- asset prices always positive (non-negative);
- Markow (reflects all information);
- non-smooth almost everywhere (non-differentiable with respect to time, because of continuous changes);
- continuous (no jumps, stock crashes).

Questions naturally arising:

Q: Can we derive a Brownian Motion?

A: Yes, this is beyond the scope of this course, but we can think of it as coming from a discrete-time random walk

$$z_{t+h} = z_t + \varepsilon_{t+h}$$

when we let $h \to 0$.

Q: Why is the change in the standard Brownian motion

$$dz = \varepsilon \sqrt{dt}$$

proportional to the square root of the time interval?

A: It needs to be proportional to assure that in the limit as $h \to 0$ the process still contains uncertainty and that the variance does not explode (i.e. it goes to infinity when $h \to 0$).

Q: Why do we not write it as a time derivative? i.e as

$$\frac{ds}{dt} = \mu s + \sigma s \frac{dz}{dt}$$

A: The problem is that dz/dt does not exist; z is almost nowhere differentiable, it is not smooth, since the process moves almost everywhere.

Q: Why are μ and σ constant?

A: Well, they do not have to be. They are constant for the GBM, but we could generally have a process like

$$ds = \mu(t,s)dt + \sigma(t,s)dz$$

This process is called an Ito process but we will stick to be GBM in this course (Black-Scholes formula is based on a GBM).

Q: What are the implications from having proportionality for the unit of time?

A: In doing so we have *ruled out jumps* in the process; the Brownian Motion is continuous. We rule out things like:

Figure 13.3

Q: Why is ε standard normal? i.e.

$$E[\varepsilon]=0;\quad var(\varepsilon)=1$$

A: The *standardization* is not a problem since we can always renormalize μ and σ.
The *normal distribution* comes from letting $h \to 0$. When we do so we increase the trading periods to infinite, so the number of random draws becomes infinite. Then we apply the Central Limit theorem and ε becomes necessarily normally distributed.

Q: Isn't the Brownian Motion a physical process?

A: Yes it is, but it was applied in financial economics first. The French mathematician Louis Bachelier developed it's mathematics in studying option pricing in his *Theorie de la Speculation* (1900). This was five years before Einstein's development of the Brownian Motion. Samuelson (1972) says on comparing Bachelier's and Einstein's derivations:

> "But years ago when I compared the two texts, I formed the judgment (which I have not checked back on) that Bachelier's methods dominated Einstein's in every element of the vector. Thus, the Einstein-Fokker-Planck Fourier equation for diffusion probabilities is already in Bachelier, along with the subtle uses of the now-standard method of reflected images."

Q: If share-prices follow Brownian motions, what *distributions* do they have.
A: Showing this rigorously is beyond the scope of this course, but it turns out that the GBM can be written as

$$d\,lns=\left(\mu-\frac{\sigma^2}{2}\right)dt+\sigma\,dz$$

This implies that the logarithms of share prices are normally distributed, which implies that the share prices are *lognormally distributed*. This is an attractive property since the share prices then cannot turn negative.

Figure 13.4

13.2 FORMING HEDGE PORTFOLIOS

Option pricing is based on *no-arbitrage* arguments.

If we can form a portfolio which is (instantaneous) risk free, the portfolio must earn the (instantaneous) risk-free rate of interest. Otherwise we could sell (or buy) the portfolio and buy (or sell) the option, and thereby making risk-free profit. This arbitrage opportunity must be ruled out in equilibrium (law of one price). This is the same argument we used in deriving the APT in Lecture 9.

Consider writing (selling) one call option and buying Δ shares (i.e. Δ is the number of shares bought). The value of such a portfolio (V_h) is

$$V_h = -c + \Delta \cdot S$$

The change in value during a "short" interval of time is

$$dV_h = -dc + \Delta \cdot dS$$

$$= -dc + \Delta \cdot \mu S dt + \Delta \cdot \sigma S dz$$

Notice that the change in the value of the portfolio is generally uncertain since dz appears in the equation. However, the value of c is also stochastic, so we can probably choose Δ in such a way that we neutralise dz. To do so we need to know how c changes over time. The value of a call-option, c, must be a function of the value of the share, S, and of time t (and also of the exercise price X, but this is constant during the life of the option):

$$c = c(S,t)$$

We need to take the differential of c, however dc <u>is not</u> $\frac{\partial c}{\partial S} dS + \frac{\partial c}{\partial t} dt$. The reason is that S is a stochastic process, so some of the second-order terms do not vanish in the differential. We need to apply Ito's Lemma in order to take the differential of c.

Ito's Lemma (Ito 1951)

Consider the Brownian Motion, where a and b are some functions of S and time t

$$dS = a(S,t)dt + b(S,t)dz$$

Consider a function c of S and of time t

$$c = c(S,t)$$

The differential of the function c is

$$dc = \left(\frac{\partial c}{\partial S} a + \frac{\partial c}{\partial t} + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} b^2 \right) dt + \frac{\partial c}{\partial S} b dz$$

Applying Ito's Lemma to our case when $a=\mu S$ and $b=\sigma S$ gives the differential of $c=c(S,t)$ as

$$dc = \left(\frac{\partial c}{\partial S}\mu S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial c}{\partial S}\sigma S dz$$

substituting this into the equation for dV_h gives

$$dV_h = -\left(\frac{\partial c}{\partial S}\mu S + \frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right)dt - \frac{\partial c}{\partial S}\sigma S dz + \Delta\cdot\mu S dt + \Delta\cdot\sigma S dz$$

Notice that if we choose Δ equal to the derivative of c with respect to S (i.e. $\Delta = \frac{\partial c}{\partial S}$) the hedge portfolio is instantaneous risk free (since dz is neutralised). Then we have

$$dV_h = \left(\Delta\cdot\mu S - \frac{\partial c}{\partial S}\mu S - \frac{\partial c}{\partial t} - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right)dt$$

but remember how we chose Δ, so

$$dV_h = -\left(\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right)dt$$

The change in value of a risk-free asset is $dV_f=r_fV_fdt$, so to have no arbitrage we must have

$$dV_h=r_fV_hdt$$

i.e.

$$r_fV_h = -\left(\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right)dt$$

Remember from the way we constructed the hedge portfolio $V_h=-c+\Delta S$, so we have

$$-r_fc + \frac{\partial c}{\partial S}Sr_f = -\left(\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right)$$

This is a second order partial differential equation, which can be solved if we have two border conditions. In this case we have two border conditions. We know that at exercise date the value of c is max{S-X,0}, and that if the share price reaches zero the share price will stay at zero (can be seen from the Geometric Brownian Motion), so the option must be worthless, i.e. $c(0,t) = 0$.

The solution to the differential equation takes the following form:

$$c = SN(d_1) - Xe^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

T-t is time left to expiration date, and

N(.) is the cumulative probability distribution function for a standardized normal variable.

13.3 BLACK-SCHOLES OPTION PRICING

The Black-Scholes formula is based on the assumption that the prices of the underlying assets change *continuously*. Here is the full set of assumptions:

- The underlying share pays no cash dividends during the life of the option.
- The option can be exercised only on the maturity date.
- There are no margin requirements, taxes, or transactions costs.
- The risk-free interest is constant.
- The volatility of the share is constant.
- The share-price follows a Geometric Brownian Motion, so only small changes in price can occur in a very short period of time.

> **Exercise 13.1** According to the formula, on which variables does the value of the call depend on? *How* does the value of the call depend on these variables? (E.g. does the value of the call increase or decrease in the interest rate?). Compare your answers with the table in Example 7.1.

Option delta

The *option delta* tells how much the value of the call changes when the price of the underlying asset changes, and is equal to $N(d_1)$ in the Black-Scholes formula:

$$\Delta = \frac{\partial C}{\partial S} = N(d_1)$$

Example 13.1

What is the price of a European call option on a non-dividend paying share when the share price is £52, the exercise price is £50, the risk-free interest rate is 5% (per year), the volatility is 30% (per year), and the time to maturity is 3 months?

We directly find S=52, X=50, σ=0.3. We need to transform the interest rate into continuous compounding, so $r = \ln(1+R) = \ln(1+0.05) = 0.0487901$. We need to transform calendar time into fractions of a year, so 3 months correspond to a quarter of a year, that is T-t=1/4. Now we calculate d_1 and d_2:

$$d_1 = \frac{\ln\left(\frac{52}{50}\right) + \left(0.0487901 + \frac{0.3^2}{2}\right)\frac{1}{4}}{0.3\sqrt{1/4}} \approx \mathbf{0.4177882}$$

$$d_2 = 0.4177882 - 0.3\sqrt{1/4} \approx \mathbf{0.2677882}$$

We need to find the values of the cumulative probability that a standard normal variable falls below d_1 and d_2. We do so by reading the table for $N(x)$ (by the end of these notes). First we find $N(d_1)$. Since d_1=0.4177882 we find the value as follows:

$$\begin{aligned} N(0.4177882) &= N(0.41) + 0.77882[N(0.42) - N(0.41)] \\ &= 0.6591 + 0.77882\times(0.6628 - 0.6591) \\ &= 0.6619816 \end{aligned}$$

and for d_2

$$\begin{aligned} N(0.2677882) &= N(0.26) + 0.77882[N(0.27) - N(0.26)] \\ &= 0.6026 + 0.77882\times(0.6064 - 0.6026) \\ &= 0.6055595 \end{aligned}$$

Now back to the formula.

Since $e^{-r(T-t)} = e^{-0.0487901/4} = 0.987877$, we have

$$\begin{aligned} c &= 52\times0.6619816 - 50\times0.987877\times0.6055595 \\ &= 34.423043 - 29.910915 \\ &= 4.512128 \end{aligned}$$

So the value of the call is **£4.51**.

Exercise 13.2 What is the value of a European call option on a non-dividend paying share when the share price is £60, the exercise price is £45, the risk-free interest rate is 7% (per year), the volatility is 50% (per year), and the time to maturity is 6 months?

13.4 THE PUT-CALL PARITY IN CONTINUOUS TIME

Consider the following two investment strategies:
(1) buying one call option with exercise price X and investing $Xe^{-r(T-t)}$ in treasury bills.
(2) buying one put option with exercise price X and buying one share (the underlying asset).

Follow the steps in Lecture 12 to obtain the Put-Call Parity in continuous time:

$$C + Xe^{-r(T-t)} = P + S$$

Example 13.2

What is the price of a European put option on a non-dividend paying share when the share price is £52, the exercise price is £50, the risk-free interest rate is 5% (per year), the volatility is 30% (per year), and the time to maturity is 3 months?

We have already calculated the price of a call-option with S=52, X=50, σ=0.3, when R=0.05. Now we can simply apply the put-call parity.
Since $e^{-r(T-t)} = e^{-0.0487901/4} = 0.987877$, we have

$$4.51 + 50\text{x}0.987877 = P + 52$$

which gives P = **£1.90**.

So the put-call parity is very useful in pricing put options if we already know the price of the call option.

Exercise 13.3 What is the value of a European put option on a non-dividend paying share when the share price is £60, the exercise price is £45, the risk-free interest rate is 7% (per year), the volatility is 50% (per year), and the time to maturity is 6 months?

13.5 HOW ACCURATE IS BLACK-SCHOLES FORMULA?

Unfortunately it does not price entirely correctly. One of the reasons is the underlying stochastic process: a Brownian Motion does not allow for "jumps", like stock crashes.

Other reasons, of course, are violation of the "standard" assumptions, like no transactions costs, no borrowing-lending constraints, etc.

REFERENCES

Copeland, Thomas E., and J. Fred Weston, *Financial Theory and Corporate Policy*, Addison-Wesley, Chapter 8, (parts E-F, H).

The following reference is optional:

Hull, John, (1997), *Options, Futures, and Other Derivatives*, Prentice-Hall, chapters (10)-11.