The Aggregate Elasticity of Factor Substitution with Middle Products

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The elasticity of substitution between productive factors is a crucial concept in the micro-economic theory of production. If capital and labor are the two inputs producing a single commodity, a change in the market returns to labor and to capital will typically encourage cost minimizing competitive firms to alter the choice of techniques – adapting factor input proportions because a reduction in wage rates relative to capital rentals provides the motivation to switch to more labor intensive modes of production. In seeking for an aggregate elasticity some sort of averaging is required when there is more than a single output and technologies differ from product to product. Whereas for each process it is possible to take the factor price changes as exogenous and have the elasticity of substitution in that sector relate the percentage increase in the labor/capital ratio used that is stimulated by a one percent decrease in the wage/rental ratio, a more aggregative measure turns the cause and effect sequence around in relating an exogenous change in the economy's endowment labor/capital ratio to the subsequent change in the wage/rental ratio.

In the search for an aggregate measure of the elasticity of substitution it is necessary to enquire as to what happens to commodity prices since they may affect factor prices. The formal setting in which the aggregation question was raised, in Jones (1965), considered a *closed* economy, one in which local demand behavior (assumed to be homothetic) helps to establish the new set of relative commodity prices when endowment proportions are altered. The aggregate elasticity of factor substitution thus combined the extent to which producers substitute labor for capital in various production processes with the indirect substitution that takes place when consumers switch demand towards more labor-intensive commodities. This suggests that for an economy engaged in international trade, especially for a small economy that is a price taker on world markets, local demand behavior has no role to play. The purpose of this paper is to enquire as to the meaning of an aggregate elasticity of factor substitution in a setting in which a small country does engage in international trade, but such trade takes place only in the *middle* of the production process so that local demand conditions do have an influence on commodity prices.

1. The Middle Products Model

The observation that much of the composition of international trade seemed to be in non-final consumer goods – i.e. in goods in process, intermediate goods, and raw materials (as well as capital goods) - prompted Kalyan Sanyal and me (1982) to cooperate on a new theoretical model of international trade. Goods such as these had two roles to play for a country assumed to be too small to affect world prices for traded goods. On the one hand, the country used some of its resources (specific factors, e.g. capital or land, and mobile labor) to produce goods *for* the world market in the *Input Tier* of the economy. On the other hand, the country made use of world trade to exchange the bundle of traded goods produced in the Input Tier for a value-equivalent bundle of goods *from* the world market, goods that would be specifically used in combination with local resources (labor) to produce commodities for local consumers in what we call the *Output Tier* of the economy. International trade takes place in the middle of the production spectrum.

Although the production structure in the Output Tier is envisaged to be that of the specific-factors variety, with each final consumption good produced with mobile labor and a specified bundle of middle products, the existence of international trade at given world prices allows an aggregation of these traded middle products into a Hicksian composite. In such a fashion, the Output Tier of the economy uses only two factors of production, labor and traded middle products. Whereas in work popularized by Neary (1978) and others, specific-factor models can be viewed as short-run versions of Heckscher-Ohlin models because the passage of time allows the conversion of one kind of input into another, such a transformation here takes place because of international trade.[1]

This paper examines a middle products model in which two final commodities are produced in the Output Tier by the use of two mobile factors – labor and traded middle products. In the Input Tier two specific types of capital are fixed in supply, and each is used with mobile labor to produce two middle products (we call them *A* and *B*) that have world markets. A special assumption is made, discussed below, that ensures that the rental on each type of capital changes by the same relative amount when factor markets are disturbed. The economy, initially in equilibrium and facing given world prices for traded middle products, experiences an increase in its endowment supply of labor. The increase in the economy's labor supply flows partly into the Output Tier and partly into the Input Tier. The basic question raised by this paper concerns the aggregate elasticity of factor substitution in the Middle Products Model by relating the relative change in the labor supply (with fixed capital stocks) to the relative change in the wage/rental ratio –

[1] Doug Purvis responded to our modeling in the Output Tier by saying that Heckscher-Ohlin does not explain trade, trade explains Heckscher-Ohlin.

with the latter in the denominator and the (negative of the) labor supply change in the numerator. Furthermore, such an aggregate elasticity is related to the aggregate substitution elasticities that can be defined separately for each of the two Tiers of the economy.

2. The Output Tier of the Economy

The total labor supply is split between Tiers. Denote that part used in the Output Tier by L_O and that used in the Input Tier by L_I. The Output Tier produces a pair of commodities: Commodity *1* is produced using labor and the amount, *A*, of a specific middle product obtained in world markets. With a_{ij} denoting an input/output coefficient, define $p_A a_{A1}$ as a_{T1}, the dollar value of traded middle products (all *A*) used to produce a unit of the first commodity. Similarly, a_{T2} is defined as $p_B a_{B2}$. In this manner the specific-factors structure in the Output Tier is transformed into a two-factor (L_O and T_O), two-commodity (x_1 and x_2) Heckscher-Ohlin model. With the price of traded middle products constant throughout, units of *A* and *B* can be selected so that p_T, p_A, and p_B are all unity.

Given these preliminaries, and letting $\hat{x}$ denote the relative change in *x*, the aggregate elasticity of substitution in the Output Tier can be defined as Φ_O:

$$(1) \qquad \Phi_O \equiv -\,(\hat{L}_O - \hat{T}_O)\,/\,(\hat{w} - \hat{p}_T)\,,$$

where, of course, $\hat{p}_T$ vanishes. Assuming balanced trade, T_O equals the value of traded middle products produced in the Input Tier, and this will change endogenously when the economy's labor force expands.

The pair of full-employment conditions in the Output Tier can be written as:

(2) $a_{L1}x_1 + a_{L2}x_2 = L_O$

(3) $a_{T1}x_1 + a_{T2}x_2 = T_O$

Following the procedure found in Jones (1965), this pair of full-employment conditions can be differentiated, with both L_O and T_O endogenous, to yield:

(4) $\lambda_{L1}\hat{x}_1 + \lambda_{L2}\hat{x}_2 = \hat{L}_O - \{\lambda_{L1}\hat{a}_{L1} + \lambda_{L2}\hat{a}_{L2}\}$

(5) $\lambda_{T1}\hat{x}_1 + \lambda_{T2}\hat{x}_2 = \hat{T}_O - \{\lambda_{T1}\hat{a}_{T1} + \lambda_{T2}\hat{a}_{T2}\}$

The λ_{ij} coefficients indicate the fraction of the Output Tier's supply of factor *i* used in the production of final commodity *j*.

The changes in input-output techniques depend upon changes in factor prices. Consider those in the first industry. Cost minimization by competitive firms ensures that:

(6) $\theta_{L1}\hat{a}_{L1} + \theta_{T1}\hat{a}_{T1} = 0,$

where the coefficient, θ_{i1}, indicates factor *i*'s distributive share in producing the first commodity. The second relationship used to solve for the change in techniques in the first industry is given by the definition of the elasticity of substitution between labor and middle products, simplified since there is no change in middle-product prices:

(7) $-\hat{a}_{L1} + \hat{a}_{T1} = \sigma_1\hat{w}$

Equations (6) and (7) can be solved for the separate changes:

$$\hat{a}_{L1} = -\theta_{T1}\sigma_1\hat{w} \quad \text{and} \quad \hat{a}_{T1} = \theta_{L1}\sigma_1\hat{w}$$

With these solutions in hand, together with the similar ones for technique changes in the second industry, equations (4) and (5) simplify to (4') and (5'):

(4') $\lambda_{L1}\hat{x}_1 + \lambda_{L2}\hat{x}_2 = \hat{L}_O + \delta_L\hat{w}$

(5’) $\lambda_{T1}\hat{x}_1 + \lambda_{T2}\hat{x}_2 = \hat{T}_O - \delta_T\ \hat{w}$

In these δ_L, for example, is the expression $\{\lambda_{L1}\theta_{T1}\sigma_1 + \lambda_{L2}\theta_{T2}\sigma_2\}$. Of special interest is the change in the ratio of the two outputs produced, $(\hat{x}_1 - \hat{x}_2)$, obtained by subtracting equation (5’) from (4’):

(8) $(\hat{x}_1 - \hat{x}_2) = (\hat{L}_O - \hat{T}_O) / |\lambda| + \{(\delta_L + \delta_T) / |\lambda|\}\ \hat{w}$

The determinant of coefficients, $|\lambda|$, is expressed also as $(\lambda_{L1} - \lambda_{T1})$, and will be a positive fraction if, as I assume, the first sector is labor-intensive, using a greater fraction of the L_O labor force than it does of the supply of middle products available in the Output Tier.

To simplify matters on the demand side, assume homotheticity, and define the elasticity of substitution in demand as:

(9) $\sigma_D \equiv -(\hat{x}_1 - \hat{x}_2) / (\hat{p}_1 - \hat{p}_2)$

The change in the wage rate (or the w/p_T ratio) is connected to the change in the commodity price ratio by the competitive profit equations of change:

$$\theta_{L1}\hat{w} = \hat{p}_1 \quad \text{and} \quad \theta_{L2}\ \hat{w} = \hat{p}_2$$

By subtraction,

(10) $|\theta|\ \hat{w} = (\hat{p}_1 - \hat{p}_2)$

The determinant, $|\theta|$, is the difference between labor’s distributive share in the first sector and its share in the second, and is a positive fraction if, as assumed, the first sector is labor intensive. Substitute this value into equation (9) and then equate the relative change in production given by (8) with the relative change in demand shown by (9) to obtain:

(11) $\{\delta_L + \delta_T + |\lambda||\theta|\ \sigma_D\}\ \hat{w} = -(\hat{L}_O - \hat{T}_O)$

The coefficient of $\hat{w}$ is the elasticity of substitution between factors in the Output Tier, as defined by equation (1). Because the δ_i's are linear in the production elasticities of substitution, σ_O is a linear expression in the three ways in which labor can substitute for capital in the Output Tier: Directly in each sector (via σ_1 and σ_2), and indirectly as consumers substitute the labor-intensive good for the middle-product intensive good in consumption (via σ_D). Indeed, σ_O is a positive weighted average of these elasticities.

The expression for σ_O matches that for the aggregate elasticity of substitution in Jones (1965). However, the setting differs in that in the earlier usage the endowment ratio whose change brought about alterations in the factor price ratio was assumed to be exogenously given. Here the exogenous change to the economy is the overall labor supply, part of it escaping to the Input Tier of the economy, allowing a greater production of middle products that can be traded for more of the appropriate balance of middle products to be used in the Output Tier, $T_{O.}$ In the Jones (1965) treatment of the aggregate elasticity of substitution, the economy was closed to international trade. Here international trade is crucial for the economy, but the nation's own demand behavior nonetheless plays an important role in the determination of the commodity price ratio for consumables and, in the 2x2 setting of the Output Tier, in the determination of the wage rate (relative to given prices for middle products).

3. The Input Tier of the Economy

The Input Tier of the economy is different from the Output Tier in that internationally traded middle products are produced in the Input Tier whereas a different bundle of middle products are used as inputs in the Output Tier. For a small economy in world markets middle product prices are taken as given, so that no demand behavior for the outputs of the Input Tier need be considered. Although both tiers of the economy are characterized by specific factors – capital, say, in the Input Tier and middle products in the Output Tier – in the Input Tier the specific factors cannot in general be joined (as they are by trade in the Output Tier). Therefore there would be two separate changes in returns to capital to be considered when the economy's labor supply expands and affects factor returns. To overcome this difficulty I assume a parity in the technology of the two sectors in the Input Tier in that the two elasticities of substitution between capital and labor are assumed to be identical, and, as well, the share of capital in each of these sectors. As a consequence, any increase in the wage rate (as of constant middle product prices) reduces the return to capital in each sector by the same relative extent, $\hat{r}$. Thus with both specific capitals fixed, the definition of the elasticity of substitution between the two inputs in the Input Tier, σ_I, can be defined as in equation (12):[2]

$$(12) \qquad \sigma_I \equiv -\hat{L}_I / (\hat{w} - \hat{r})$$

In specific-factors models with labor mobile and specific capital fixed, the elasticity that is usually used to capture technology is the elasticity of demand for labor – the relative increase in the labor used in sector *i* when the wage rate relative to output price (here given) falls by one percent. This is γ_i, equal to $-\hat{L}_i / \hat{w}$. But since the competitive profit equations of change are the same in the two sectors of the Input Tier (because the

[2] It is easy to show that any expansion of the labor force in the Input Tier does not alter the ratio of outputs of the two middle products produced there if, as assumed, both elasticities of substitution and capital shares are assumed the same in the two sectors. Details for the specific-factors model are provided in Jones (1971).

two capital shares are assumed equal, the same as θ_{KI}, the share of capital in the Input Tier),

$$\theta_{LI}\ \hat{w}\ +\ \theta_{KI}\ \hat{r}\ =\ 0,$$

which implies that $\hat{w}$ equals $\theta_{KI}\,(\hat{w} - \hat{r})$. Thus γ_i equals the common σ_i in the Input Tier divided by θ_{KI}. It follows by equation (12) that the common σ_i in the Input Tier is the same as σ_I, the aggregate elasticity of substitution for the Input Tier.

4. The Aggregate Elasticity of Substitution for the Economy

So far what has been provided are the appropriately defined sub-aggregates of the elasticity of substitution for each Tier. What has been left out is the division of the economy's increased labor supply between Tiers and how this affects aggregation for the economy.

Consider the change in the wage rate discussed for each Tier of the economy. By the definition of σ_O in equation (1), with fixed prices for traded middle products,

$$\hat{w}\ =\ -\ (\hat{L}_O - \hat{T}_O)\ /\ \sigma_O$$

In the Input Tier the relative change in the wage rate is shown by

$$\hat{w}\ =\ -\ (\theta_{KI}\ /\ \sigma_I)\ \hat{L}_I$$

Equating these two values for the change in the relative wage rate yields equation (13):

$$(13)\qquad (\hat{L}_O - \hat{T}_O)\ =\ [\theta_{KI}\sigma_O/\sigma_I]\ \hat{L}_I$$

The value of the output produced in the Input Tier is T_I, and with balanced trade this is equal to the value of middle product *inputs* to the Output Tier, T_O. In the Input Tier capital stocks are being held constant, so that

$$\hat{T}_I = [dT_I/dL_I]\,[L_I/T_I]\,\hat{L}_I$$

With the wage rate, *w*, matched by labor's marginal product, dT_I/dL_I, and the distributive share of labor in the Input Tier, θ_{LI}, captured by $[wL_I/T_I]$, $\hat{T}_O$ is expressed in (14):

(14) $$\hat{T}_O = \theta_{LI}\,\hat{L}_I,$$

Labor's share in the Input Tier captures the productivity of the new flow of labor into the Input Tier. Thus ($\hat{L}_O - \hat{T}_O$) equals ($\hat{L}_O - \theta_{LI}\,\hat{L}_I$), and, using (13), yields equation (15) as one relationship between the changed labor flows into each tier:

(15) $$-\{\theta_{KI}\sigma_O/\sigma_I + \theta_{LI}\}\,\hat{L}_I + \hat{L}_O = 0$$

All labor is fully employed so that $(L_O + L_I) = L$. Taking relative changes,

(16) $$\lambda_{LI}\,\hat{L}_I + \lambda_{LO}\,\hat{L}_O = \hat{L}$$

Multiply all terms in (15) by λ_{LO} and subtract from equation (16) to obtain:

(17) $$\{\lambda_{LI} + \lambda_{LO}\,[\theta_{KI}\sigma_O/\sigma_I + \theta_{LI}]\}\,\hat{L}_I = \hat{L}$$

I have saved the unadorned term, σ, to denote the aggregate elasticity of substitution between labor and capital in the Middle Products Model. With capital fixed,

(18) $$\sigma \equiv -\,\hat{L}\,/\,(\hat{w} - \hat{r})$$

Divide both sides of equation (17) by $-(\hat{w} - \hat{r})$, recalling that $(\hat{w} - \hat{r})$ equals $(\hat{w}\,/\,\theta_{KI})$, to obtain equation (19) as the general decomposition of the economy-wide elasticity of substitution between labor and capital in terms of the sub-aggregates in each of the pair of tiers in the economy:

(19) $$\sigma = \lambda_{LI}\,\sigma_I + \lambda_{LO}\,[\theta_{LI}\,\sigma_I + \theta_{KI}\,\sigma_O]$$

The last term in brackets is an average of the two sub-aggregates, and the entire expression is a weighted average of the bracketed term and the elasticity of substitution in the Input Tier, with the weights represented by the division of labor between the two tiers of the economy. The overall elasticity of substitution for the Middle Products Model must lie strictly between the sub-aggregate elasticity in the Input and Output Tiers of the economy.

5. Concluding Remarks

The elasticity of substitution in any one sector reveals how a change in the input price vector induces a change in the ratio of inputs in production. The idea that factor prices and factor proportions are related can also be suggested for the reverse order of causation: a change in relative factor proportions brings about a change in factor prices. This latter conception is especially attractive in considering factor price and quantity relationships for the entire economy, including the notion that factors may be substituted for each other indirectly when a change in relative commodity prices motivates consumers to alter their demands for commodities with different factor proportions and thus to put further pressure for factor prices to change.

For a simple two by two general equilibrium model of a closed economy, such an aggregate expression for the elasticity of substitution was developed in Jones (1965). The question that can then be asked is whether this same procedure can be utilized if an economy is engaged in international trade. The problem is that the usual depiction of such an economy has relative prices of final consumer goods determined in world markets, with world supplies as well as the pattern of world demand affecting the outcome. Such a problem is avoided in the model of trade in Middle Products developed in Sanyal and Jones (1982). International trade does take place, but not for the commodities appearing in final form for consumption. Instead, international trade allows a small price-taking country the possibility of exchanging items it produces for the world market for a different bundle of items it requires as inputs into the final stages of the production process where local inputs (labor) are combined with inputs obtained from the world market to produce final non-traded commodities for local consumers. There emerges an expression for the economy-wide aggregate elasticity of substitution between labor and capital for an economy engaged in trade that lies strictly between well-defined elasticities of substitution in each tier of the economy – the Input Tier in which capital and labor produce goods for the world market, and an Output Tier in which labor is combined with a different array of traded middle products to produce commodities for local consumption.

References:

Jones, Ronald W. (1965): "The Structure of Simple General Equilibrium Models," *Journal of Political Economy,* v. 63, pp. 557-72.

________________(1971): "A Three-Factor Model in Theory, Trade and History," Ch. 1 in Bhagwati, Jones, Mundell and Vanek (eds.), *Trade, Balance of Payments and Growth* (North-Holland, Amsterdam).

Neary, J. Peter (1978): "Short-run Capital Specificity and the Pure Theory of International Trade," *Economic Journal,* v. 88, pp. 488-510.

Sanyal, Kalyan and Ronald W. Jones (1982): "The Theory of Trade in Middle Products," *American Economic Review*, v. 72, pp. 16-31.