1 Appendix C

In this Appendix we formally study the model where the k bidders also care about the payment consequences of their bidding. The utility of each bidder is $10-t$ if he receives the object and pays amount t and 0 if he does not receive the object (and therefore pays zero). Each bidder maximizes his expected utility. Finally, we assume that the highest allowed bid is 1 and characterize the limit of equilibria as the bid increment becomes small, i.e. $\varepsilon \to 0$. Before proceeding we introduce some notations. For any $\varepsilon > 0$, bid b_r with $r = 1, 2, ..., \frac{1}{\varepsilon} + 1$ denotes a bid with which the winner needs to pay $1 - \varepsilon(r-1)$. The probability of bidding b_r is denoted by p_r.

We characterize the equilibrium in steps.

Step 1: It is easy to see that for each $\varepsilon > 0$ there is a number $l > 0$, such that $p_1, .., p_l > 0$ and $p_{l+1} = 0$ and by construction l is bounded for each $\varepsilon > 0$. First, we consider the case where $\lim\limits_{\varepsilon \to 0} l < \infty$. The other case is revisited at the end of Step 5.

Since l remains finite as $\varepsilon \to 0$ the payment remains 1 no matter whether bid b_1 or b_l is used or any bid in between. Therefore, the probability of winning when using any of those bids is equal to each other.

Step 2: Suppose that in equilibrium all bidders use only those l bids. As we argued in Step 1, this implies that the probability of winning when using any of those bids is equal to each other. Therefore, the equilibrium bidding strategies also constitute an equilibrium of the baseline model of probability maximization.

For this to be the case there can be no bid less than bid b_l that provides a profitable deviation. To determine whether this is indeed the case, it is sufficient to establish that bidding zero does not provide a profitable deviation. The payoff from bidding zero is $U(0) = 10tie$, where tie denotes the probability that all other bidders tie. Since the probability of winning when using any of the highest l bids is the same, this probability must be equal to $1/k$, because we study symmetric equilibria only. Therefore, the utility from bidding one of those l high bids is $9\frac{1}{k}$. If $9\frac{1}{k} \geq 10tie$ holds, then there is an equilibrium where only the l highest bids are used and all bids yields a probability of winning, which is equal to $1/k$.

Step 3: Take now the other case, i.e. suppose that

$$10tie > \frac{9}{k}. \tag{1}$$

In this case, the equilibrium of the baseline model is not immune to a deviation to bid zero, when the payment consequences of bidding are taken into account. It is an open question whether inequality (1) holds or fails for most values of k. However, if there are only a few bidders ($k = 4, 5, 6, 7$) one can indeed show that it holds and therefore the equilibrium of the baseline model is not immune to low bids.[1]

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[1] For example the equilibrium of the baseline model when $n = 5$ is such that $tie = 0.199 > \frac{9}{5*10} = 0.18$.

In this case there must be a $h > 0$ such that $\lim\limits_{\varepsilon \to 0}(p_{h/\varepsilon}+p_{1+h/\varepsilon}+...+p_{1+1/\varepsilon}) > 0$ and $\lim\limits_{\varepsilon \to 0}(p_{l+1}+p_{l+2}+...+p_{-1+h/\varepsilon}) = 0$. Take the smallest such h, and note that by construction $b_{h/\varepsilon}$ is a bid that means a payment $1-(h-\varepsilon)$. Then it is easy to establish that $\lim\limits_{\varepsilon \to 0} p_{h/\varepsilon} = 0$, since otherwise using bid $b_{-1+h/\varepsilon}$ is a profitable deviation if ε is close enough to zero.[2] By using a similar argument as in Theorem 1, one can show that for all $m > h/\varepsilon$ it holds that $\lim\limits_{\varepsilon \to 0} p_{m-1} \geq \lim\limits_{\varepsilon \to 0} p_m$ and therefore for all $m \geq h/\varepsilon$ it holds that $\lim\limits_{\varepsilon \to 0} p_m = 0$.

Step 4: One can also show that for all $q \in (h,1)$ it holds that $\lim\limits_{\varepsilon \to 0}(p_{q/\varepsilon} + p_{1+q/\varepsilon} + ... + p_{1+1/\varepsilon}) > 0$. Otherwise, $b_{1+q/\varepsilon} \to 1-q$ would be the lowest bid that is in the support of equilibrium strategies. Bidding $1-q$ and bidding 0 would then yield the same winning probability, since as we argued in Step 3 $\lim\limits_{\varepsilon \to 0} p_{1+q/\varepsilon} = 0$ and thus a zero bid would win (almost surely) whenever a bid $1-q$ wins. But then bidding $1-q$ would not be optimal in the limit. A similar argument shows that for any $r > q > h$ it holds that $\lim\limits_{\varepsilon \to 0}(p_{q/\varepsilon} + p_{1+q/\varepsilon} + ... + p_{r/\varepsilon}) > 0$ and thus there is no gap in the support of equilibrium bids in the limit.

Step 5: Now, we show that in the limit it has a zero probability that two bidders tie at a bid level greater than l. For this it is sufficient to show that $p^2_{h/\varepsilon}+p^2_{1+h/\varepsilon}+...+p^2_{1+1/\varepsilon} \to 0$ as $\varepsilon \to 0$. In turn, for this it is sufficient to obtain an upper bound $r \in \mathbb{R}^+$, s.t. for all $s \geq h$ it holds that $p_{s/\varepsilon} \leq r\varepsilon$ if ε is close enough to 0. Suppose that this does not hold and there exists an s such that such an upper bound cannot be found. Then in the limit as $\varepsilon \to 0$, the density of equilibrium bids approach infinity at a bid equal to $b_{s/\varepsilon} = 1-s+\varepsilon \to 1-s$. But standard auction theoretic arguments imply that then every bidder would want to bid slightly above $1-s$ instead of bidding $1-s$, which leads to a contradiction that such a bid is optimal.

Let us revisit now the case where $\lim\limits_{\varepsilon \to 0} l = \infty$. In this case, a similar argument to the one in Step 4 shows that in the limiting bid function has no gap and its support goes all the way down to a bid $b = 0$. It is also easy to show, that the limiting bid function has an atom only at the highest bid $b = 1$, otherwise standard auction theoretic arguments show that the bidders would have an incentive to bid slightly above this atom. Moreover, the argument of Step 5 also goes through and therefore the probability of tie at a bid level lower than the highest allowed bid, $b = 1$ goes to zero as $\varepsilon \to 0$. The upper bound provided in Step 6 remains valid in this case for P_k as well. In other words, the analysis does not change in any important way.[3]

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[2] The intuition is straightforward: by just bidding ε higher a bidder can increase his probability of winning by a positive amount if $\lim\limits_{\varepsilon \to 0} p_{k/\varepsilon} > 0$, which cannot be optimal as we know it from standard auction theoretic considerations.

[3] It is also a strong conjecture of the authors that this case does not arise when $k \geq 4$, since it requires that infinitely many possible bids are made in the equilibrium of the baseline model. However, it is not a central question for us whether this case obtains and thus a formal proof is not attempted to rule it out.

Step 6: We would like to obtain an upper bound on the probability of bidding in the low region, $P_k = \lim_{\varepsilon \to 0} p_{h/\varepsilon} + p_{1+h/\varepsilon} + ... + p_{1+1/\varepsilon}$. Suppose that a bidder bids exactly at bid level $b_{h/\varepsilon} = 1 - h + \varepsilon \to 1 - h$. The probability of winning with such a bid is equal to the probability that the $k-1$ other bidders tie (tie) plus the probability that all others bid less than $1-h$. This probability is bounded below by the probability that neglects the case when at least two other bidders bid less than $1-h$. Therefore, the probability of winning with such a bid is

$$Pr_h \geq tie + (k-1)P_k tie_{-1},$$

where tie_{-1} is the probability that $k-2$ other bidders all use the first l bids and they all tie. The utility from bidding $1-h$ is such that

$$U_h \geq (10 - (1-h))(tie + (k-1)P_k tie_{-1}).$$

The utility from bidding 0 is

$$U(0) = 10tie.$$

Since both bids are optimal in the limit by Step 4, it holds that

$$10tie \geq (9+h)(tie + (k-1)P_k tie_{-1}).$$

Using that $h \geq 0$ it follows that

$$10tie \geq 9(tie + (k-1)P_k tie_{-1})$$

or

$$P_k \leq \frac{tie}{9(k-1)tie_{-1}}. \tag{2}$$

We show that for all k even or odd and large enough, it holds that $\frac{tie}{tie_{-1}} \leq \frac{k-1}{3}$ and thus $P_k \leq \frac{1}{27}$. We start with the case when k is even. For simplicity we work out the case when $k=6$, but a generalization is straightforward. As we argued in Step 5, the probability of a complete tie is equal to the probability of a tie at the top l bids. Therefore,

$$tie = (p_1^5 + p_2^5 + ... + p_l^5) + 10(\sum_{l \geq i > j} p_i^3 p_j^2 + \sum_{i < j \leq l} p_i^3 p_j^2).$$

Similar considerations yield that

$$tie_{-1} = (p_1^4 + p_2^4 + ... + p_l^4) + 6(\sum_{i>j} p_i^2 p_j^2 + \sum_{i<j} p_i^2 p_j^2).$$

Since $p_1 + p_2 + ... + p_l \leq 1$ it follows that

$$tie_{-1} \geq tie_{-1}(p_1 + p_2 + ... + p_l) >$$

$$> (p_1^5 + p_2^5 + ... + p_l^5) + 6(\sum_{l \geq i > j} p_i^3 p_j^2 + \sum_{i < j \leq l} p_i^3 p_j^2).$$

Therefore, $\frac{tie_{-1}}{tie} > \frac{6}{10}$. This argument can be generalized to see that for any k even it holds that $\frac{tie_{-1}}{tie} > \frac{3}{k-1}$. Then equation (2) implies that $P_k \leq \frac{1}{27}$ holds for any k even. When k is odd, this argument does not work since the term $p_1^2 p_2^2 ... p_l^2$ is not captured by $tie_{-1}(p_1 + p_2 + ... + p_l)$, however it plays a role in the expression tie. However, it is easy to see that as k becomes large, this term becomes negligible, which concludes the proof. [4]

Step 7: Finally, we are ready to calculate the equilibrium for our model with payment consequences. After conjecturing the number of bids in the high region (l) one can use the above results for this calculation. The equilibrium utility is equal to $10tie$ by Step 3. The utility upon winning with any high bid, $b \in \{b_1, b_2, .., b_l\}$ is $10-1=9$. Therefore, the unconditional utility is equal to the probability of winning with such a bid times 9. To calculate these probabilities, for $j = 1, 2, ..., l$ define W_j as the probability that no other bidder places bid b_j or a *unique* bid higher than b_j. This is the probability that a bidder becomes an outright winner if he places bid p_j. Let t_j be the probability that there is a tie among all bidders if a bidder places bid b_j. Probability t_j is equal to the probability that a complete tie occurs at bids $b_1, b_2, .., b_l$, because a tie has zero probability at lower bid levels by Step 5. Using this observation it follows that t_j is pinned down by $(p_1, p_2, .., p_l)$. Then the total probability of winning with bid b_j is equal to $\frac{W_j}{1-t_j}$ using that if there is a tie among all bidders, then the game is played again. Since both W_j and t_j are determined by $(p_1, p_2, .., p_l)$, it follows that the winning probability is determined by those probabilities as well. Since the incentive conditions $\forall j = 1, 2, ..., l$ can be written as

$$10tie = 9\frac{W_j}{1-t_j}$$

$\forall j = 1, 2, ..., l$, they are completely determined by $(p_1, p_2, .., p_l)$, since variable tie is also pinned down by $(p_1, p_2, .., p_l)$ by Step 5. Therefore, one has a system of equations containing unknowns $(p_1, p_2, .., p_l)$ only. The exact form of those equations are similar to the ones in the main text and are therefore omitted.

In the case when $k = 4$ an equilibrium is such that

$$p_1 = 0.484,\ p_2 = 0.482,\ p_3 = 0.022 \text{ and } P_4 = 0.012.$$

In the case when $k = 5$ an equilibrium is such that

$$p_1 = 0.368,\ p_2 = 0.331,\ p_3 = 0.189,\ p_4 = 0.065,\ p_5 = 0.014 \text{ and } P_5 = 0.033.$$

In the case when $k = 6$ an equilibrium is such that

$$p_1 = 0.332,\ p_2 = 0.305,\ p_3 = 0.240,\ p_4 = 0.107 \text{ and } P_6 = 0.016.$$

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[4] It is our strong conjecture that for some $\alpha > 0$ it holds that for all k, $\frac{tie}{tie_{-1}} \leq \alpha$ and thus P_k converges to zero at a rate of $\frac{1}{k}$. If one calculates the equilibrium of the model with payment consequences, then $\frac{tie}{tie_{-1}}$ for the cases of $k = 4, 5, 6, 7$ is $0.48, 1.8, 0.85$ and 1.21 respectively. Our conjecture is that this sequence converges to 1 as k becomes large.

In the case when $k = 7$ an equilibrium is such that

$$p_1 = 0.298,\ p_2 = 0.275,\ p_3 = 0.231,\ p_4 = 0.138,\ p_5 = 0.039 \text{ and } P_6 = 0.020.$$

As one can see, these probabilities are fairly similar numerically to the ones in the baseline model.