Competing Auctions: Finite Markets and Convergence*

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Abstract

The literature on competing auctions offers a model where sellers compete for buyers by setting reserve prices freely. An important outstanding conjecture (e.g. Peters and Severinov (1997)) is that the sellers post prices close to their marginal costs when the market becomes large. This conjecture is confirmed in this paper. More precisely, we show that if all sellers have zero costs, then the equilibrium reserve price converges to 0 in distribution. I also show that if there is a high enough lower bound on the buyers' valuations, then there is a symmetric pure strategy equilibrium. In this equilibrium, if the number of buyers (sellers) increases, then the equilibrium reserve price increases (decreases) and the reserve price is decreasing in the size of the market.

1 Introduction

Among others, McAfee (1993), Peters and Severinov (1997) and Burguet and Sakovics (1999) study sellers who compete by posting second price auctions setting the reserve price as they wish and the buyers decide which seller to visit given the reserve prices posted.[1] Peters and Severinov (1997) consider the case where the market is infinitely large and show that in equilibrium the sellers post reserve prices that are equal to their production costs. They also show that *if* there is an equilibrium for each finite market size where the sellers post identical (and deterministic) reserve prices, then the equilibrium reserve price converges to the cost of production, which is normalized to zero. However, as Burguet and Sakovics (1999) argue, such an equilibrium does not exist in the case when there are two sellers and one can extend their argument to the case of any (finite) number of sellers. Therefore, the equilibrium reserve price in large but finite markets is not settled by those articles. Hernando - Veciana (2005) shows that if only a finite number of reserve prices are allowed, then the equilibrium reserve price in large but finite markets converges to the cost of the sellers. While this result is interesting, it depends on the restriction on the set of admissible reserve prices.

Our paper revisits the question of convergence by providing two results. First, we show that there exist equilibria (possibly in mixed strategies) and in all equilibria the reserve price each seller posts converges to 0 in distribution (and in support) as the market becomes large. The logic is that as the market becomes large each seller loses his effect on the utility levels of the buyers and thus has limited incentives to increase his reserve price. On the other hand, by decreasing the reserve price a seller is able to attract extra visitors. Since the utility effect is small in large markets, the seller has to just provide a market utility to the buyers visiting him and can capture the surplus, which is generated beyond that utility level. This extra surplus stays positive, since each seller has finitely many expected visitors regardless of the market size, and thus an extra visitor increases the probability of sale increasing the surplus generated by the seller. As a consequence, decreasing the reserve price offers benefits at little costs to the seller as the market becomes large.

To confirm this conjecture, we need to show that as the market becomes large each seller loses his effect on the utility levels of the buyers. We consider a game with finitely many players and use properties of the binomial distribution to derive the desired result. This result has been elusive so far, since the utility of the buyers is a very complicated polynomial function of the reserve prices and a closed form characterization is not obtainable. To simplify the problem, we employ a somewhat indirect method. First, we concentrate on the highest reserve price in the support of the equilibrium. Our aim is to show that the poster of that

*I would like to thank Angel Hernando-Veciana, Michael Peters, Jozsef Sakovics and participants at the University of Rochester, the 2007 Fall Midwest Theory Conference and the 2008 Econometric Society Summer Meetings for their useful comments. All remaining errors are mine.

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[1] The environment in Peters (1991, 2000) is also similar, but he focuses on the case where sellers post prices and not auctions.

price has an incentive to reduce his reserve price regardless of what reserve price the other sellers posted, if the market is large enough. By focusing on the upper end of the support, we are able to characterize the changes in revenues in a much simpler formula, utilizing the fact that the seller posting the highest price is only visited by the buyers with the highest valuations. Second, we characterize the utility of a buyer type who does *not* visit the seller who posted the highest reserve price and thus his utility changes only because the change in the highest reserve price changes the visiting probabilities of the other buyers. The change of utility resulting from such a change in visiting probabilities is relatively easy to characterize by appealing to the envelope theorem. Third, we calculate the change of utility of buyer types who *do* visit the seller with the highest reservation price, by using the envelope theorem and the utility of a type who does *not* visit this seller. This indirect method establishes that the utility effect vanishes in large markets leading to the convergence result.

Our second result concerns cases in which a pure strategy equilibrium exist, i.e. where the sellers post non-random reserve prices, a deviation from the previous literature, which makes assumptions that preclude the existence of a pure strategy equilibrium. More precisely, the above articles assume that the lowest possible valuation of the buyers is equal to the production cost (which is normalized to zero in this paper). Then the first order condition for seller optimality suggests that the sellers choose a zero reserve price, but then it is costless to increase the reserve price, since only buyers with the lowest possible valuations (which is zero) are lost. So the second order condition fails and a pure strategy equilibrium does not exist. Therefore, we consider the case where the lowest possible valuation is positive and provide a sufficient condition for a pure strategy equilibrium to exist. We provide intuitive comparative statics results and also characterize the rate of convergence of the reserve price to zero. If the number of buyers (sellers) increases, then the equilibrium reserve price increases (decreases) and the reserve price decreases in the size of the market. The equilibrium reserve price converges to 0 at the quick rate of $\frac{1}{n}$, if the ratio of sellers to buyers is constant.

2 Model and Analysis

There are k expected revenue maximizing sellers, each with one unit of an indivisible good. There are n risk neutral buyers, each with a unit demand of the good. First, the k sellers each post a reserve price r_k and then each buyer decides which seller to visit. At seller j the buyers present engage in a second price auction with reserve price r_j.[2] Buyer i has valuation v_i which is distributed on $[a, a+1]$ according to cdf F and a strictly positive density f and is independent of the valuation of any other buyers. Furthermore, assume that the density is bounded and for all $x \in [a, a+1]$ it holds that $f(x)$.

Given that the sellers post second price auctions, it is a dominant strategy for each buyer to submit a bid equal to his valuation. Therefore, the only decision a buyer needs to make is which seller to visit. We concentrate on equilibria where the buyers employ symmetric visiting strategies. Formally, for $j = 1, 2, ..., k$ it holds that $r_j = r$ and for $i = 1, 2, ..., n$ the probability that buyer i with type v visits seller j when the reserve prices are $(r_1, r_2, ..., r_k)$ is such that $\pi_i^j(v, r_1, r_2, ..., r_k) = \pi^j(v, r_1, r_2, ..., r_k)$.

We consider the question whether the equilibria of games with a finite number of buyers and sellers converge to the equilibrium of the infinite game where all sellers post a zero reserve price. First, note that the profit functions are continuous in the reserve prices. Therefore, standard fixed point theorems can be used to establish existence of a mixed strategy equilibrium in any game with finite number of buyers and sellers. Let us start the analysis by characterizing the equilibrium visiting strategies of the buyers given the reserve prices posted by the sellers.

Lemma 1 *If $r_1 \leq r_2 \leq ... \leq r_{k-1} \leq r_k < a+1$, then in the symmetric equilibrium in the buyers' stage game there exist unique cutoff types $\max\{a, r_1\} = t_0 \leq t_1 \leq t_2 \leq .... \leq t_{k-1} \leq t_k = a+1$ such that buyer types between t_l and t_{l+1} visit the first $l+1$ sellers with probability $\frac{1}{l+1}$. Moreover, if $r_k > r_{k-1}$ then the cutoff values are differentiable in r_k and it holds that $\frac{\partial t_{k-1}}{\partial r_k} > 0$ and $\frac{\partial t_{k-2}}{\partial r_k} > 0$. Also, it holds for any $x \geq t_{k-2}$ that if r_k decreases, then $u(x)$ increases.*

Proof. See the online Appendix. ■

[2] Similarly to Burguet and Sakovics (1999), any mechanism that is efficient when the buyers are symmetric, like first-price or all-pay auctions, would yield similar result. More precisely, there would still be an equilibrium, where the sellers post reserve prices as characterized below.

While very high types (above t_{k-1}) randomize between all k sellers, lower types restrict their visits to sellers who posted lower reserve prices. Very low types (types lower than t_1) only visit the seller with the lowest reserve price. The existence of such cutoff types is proved by an extension of the two seller analysis provided by Burguet and Sakovics (1999). We establish the uniqueness of the cutoff types by making use of the envelope theorem in an iterative procedure.

We are ready to prove the convergence result discussed above. The proof follows our intuitive argument. We first show that as the market grows large, each seller loses his effect on the equilibrium utilities of the buyers. Then we conclude the proof by showing that serving an extra type has a non-vanishing positive effect on welfare, which is thus captured by the seller as revenue in the limit.

Theorem 2 *Take a sequence of games where $\lim_{l\to\infty} k^l, n^l = \infty$ and for all l it holds that $\frac{n^l}{k^l} \leq \rho$ for some $\rho > 0$. Then for any sequence of equilibria where the buyers use symmetric strategies, the reserve prices posted by the sellers converge to zero in distribution.*

Proof. Take a sequence of games characterized by the number of sellers k^l and the number of buyers n^l and assume that $k^l, n^l \to \infty$ as $l \to \infty$. Denote the support of the equilibrium strategies of seller i as $[c_i^l, d_i^l]$ and denote the density function of the equilibrium reserve prices[3] by h_i^l and the corresponding distribution function as H_i^l. Note, that it is without of loss of generality to assume that the support is a *closed* interval, i.e. it is a best response to post reserve price d_i^l, because the game between the sellers is continuous in the reserve prices.

Case 1: First, assume that the equilibrium bid distribution does not put positive probability on d_i^l, i.e. for all i and l it holds that

$$\lim_{x \nearrow d_i^l} H_i^l(x) = 1.$$

Let $m(l) \in \{1, 2, ..., k^l\}$ be such that $d_{m(l)}^l \geq d_j^l$ for all j. Then by posting a reserve price $d_{m(l)}^l$, seller $m(l)$ posts a higher reserve price than all other sellers with probability 1 and thus receives visits from buyers with type above t_{k-1}. *We show below that if seller i posts a positive reserve price r_i^l such that $r_i^l > r_j^l$ for all $j \neq i$, then he can profitably decrease his reserve price if l is large enough.* One needs to establish a uniform bound on the market size exists, that does not depend on the reserve prices posted by the other sellers. This would imply that for a large enough l it is not profitable for seller $m(l)$ to post $d_{m(l)}^l > 0$ and thus $\lim_{l\to\infty} d_{m(l)}^l \leq 0$ must hold. However, it is routine to show that posting a negative reserve price is never optimal and thus for all i it holds that

$$\lim_{l\to\infty} c_i^l = \lim_{l\to\infty} d_i^l = 0$$

as claimed.

First, let us the denote the (expected) revenue of seller k as $R_k(r_1, r_2, ..., r_k)$. Extending the argument from Burguet and Sakovics (1999), one can show that for any vector of reserve prices the left hand derivative of π_k with respect to r_k exists. Moreover, if $r_i < r_k$ then this left hand derivative is continuous in r_i. Let $r_i \leq r_j$ if $i < j$ and assume that that $r_{k-q} < r_{k-q+1} = r_k$. Then using the continuity of the left hand derivative in r_t for all $t < k - q$, it follows that it is sufficient to prove that there exists an α such that $\frac{\partial R_k}{\partial r_k} \leq \alpha < 0$ for a sufficiently large k for all vectors of reserve prices where $r_{k-q-1} < r_{k-q}$, since then continuity would imply that the derivative stays (strictly) negative when $r_{k-q-1} = r_{k-q}$. Therefore, it is sufficient to consider reserve price vectors such that $r_{k-q-1} < r_{k-q} < r_{k-q+1} = r_k$ and prove that seller k has an incentive to decrease his reserve price if $r_k > 0$ and k is large enough.

In case 1 it holds with probability 1 that $q = 1$ and thus it is sufficient to consider reserve price vectors such that $r_{k-2} < r_{k-1} < r_k$ holds. As a first step we characterize the change in utilities of the buyers who visit seller $i = m(l)$, if seller i reduces t_{k-1} slightly by reducing his reserve price slightly from d_i. Using the incentive conditions of the buyers one can establish that for all $x > t_{k-1}$ it holds that

$$0 \geq \frac{\partial u(x)}{\partial t_{k-1}} \geq -\frac{n-1}{k(k-1)} f(t_{k-1})(a+1). \tag{1}$$

[3] The language of the proof implicitly assumes that we have a mixed strategy equilibrium, but the case of pure strategies is covered in case 2 in the Appendix.

(For the proof of this result see Lemma 4 in the Appendix.)

We can now calculate the expected revenue of seller k as the total surplus generated at seller k minus the total utilities of the types visiting seller k. For all $x \geq t_{k-1}$ let $G(x)$ denote the probability that seller k sells to a buyer with type less than x or does not sell at all. This event happens if and only if no buyer with type greater than x visits seller k, and thus $G(x) = (1 - \frac{1-F(x)}{k})^n$. Letting $g(x) = \frac{\partial G}{\partial x}$, the total surplus generated at k is

$$W_k = \int_{t_{k-1}}^{a+1} x g(x) dx.$$

Since such types visit seller k with probability $1/k$ it follows that the sum of utilities generated at k can be written as

$$C_k = n \int_{t_{k-1}}^{a+1} \frac{1}{k} u(x) f(x) dx.$$

Then the expected revenue is $R_k = W_k - C_k$ and one can assume that seller k maximizes $\widehat{R}_k = \frac{k}{n} R_k$. It holds that

$$\frac{\partial \widehat{R}_k}{\partial t_{k-1}} = -f(t_{k-1})[(1 - \frac{1 - F(t_{k-1})}{k})^{n-1} t_{k-1} - u(t_{k-1})] - \int_{t_{k-1}}^{a+1} f(x) \frac{\partial u(x)}{\partial t_{k-1}} dx. \quad (2)$$

Since type t_{k-1} wins with probability $(1 - \frac{1-F(t_{k-1})}{k})^{n-1}$ and pays r_k at seller k it holds that

$$u(t_{k-1}) = (1 - \frac{1 - F(t_{k-1})}{k})^{n-1} (t_{k-1} - r_k)$$

and thus

$$(1 - \frac{1 - F(t_{k-1})}{k})^{n-1} (t_{k-1} - u(t_{k-1}) = (1 - \frac{1 - F(t_{k-1})}{k})^{n-1} r_k. \quad (3)$$

Using (1) and (3), one obtains from (2) that

$$\frac{\partial \widehat{R}_k}{\partial t_{k-1}} \leq -f(t_{k-1})(1 - \frac{1 - F(t_{k-1})}{k})^{n-1} r_k + \frac{n-1}{k(k-1)} f(t_{k-1})(a+1).$$

Inequality $\frac{n}{k} \leq \rho$ implies that for all $k \geq 2$

$$(1 - \frac{1 - F(t_{k-1})}{k})^{n-1} \geq (1 - \frac{1}{k})^{n-1} \geq (1 - \frac{1}{k})^{k\rho} \geq (1 - \frac{1}{2})^{2\rho} = (\frac{1}{4})^{\rho}.$$

Therefore, for all l such that $\frac{n}{k} \leq \rho$ (and $k \geq 2$) it holds that

$$\frac{\frac{\partial \widehat{R}_k}{\partial t_{k-1}}}{f(t_{k-1})} \leq -r_k (\frac{1}{4})^{\rho} + \frac{\rho}{(k-1)} (a+1) < 0,$$

if

$$k > \frac{\rho(a+1)}{r_k (\frac{1}{4})^{\rho}}. \quad (4)$$

Therefore, it is profitable for the seller to decrease t_{k-1}, (which is the same as decreasing r_k, because the variables are one-to-one by Lemma 1), if l is large enough so that $k > \max\{\frac{\rho}{r_k (\frac{1}{4})^{\rho}}, 2\}$. Note, that the established convergence is uniform, i.e. the required l does not depend on the reserve prices posted by the other sellers (as long as they are strictly less than r_k).

Case 2: Now, assume that the equilibrium bid distribution may put positive probability on d_i^l, i.e. it *does not* hold that

$$\lim_{l \to \infty} \lim_{x \nearrow d^l} H^l(x) = 1.$$

The proof is given in the Appendix. ■

Using the above argument one can also show that the rate of convergence is fairly quick under our assumptions. Inspecting formula (4) that is valid in case 1, suggests that the upper end of the equilibrium reserve price distribution $\overline{r}$ satisfies if $\overline{r} < \frac{D}{k}$ for some constant D. In other words, we can provide an upper bound for equilibrium reserve prices that is inversely proportional to the market size as measured by the number of buyers k. The proof for Case 2 in the Appendix shows that his result holds there as well.

It is our conjecture that the conclusion of Theorem 2 (convergence) would also hold under the weaker condition that

$$\lim_{l \to \infty} \frac{n^l - 1}{k^l (k^l - 1)} = 0,$$

but a proof is currently unavailable. We will see in the next Section that this condition is sufficient in the case when a pure strategy equilibrium exists.

It would also be interesting to consider the case where the sellers are heterogenous in their production costs. While a formal analysis is beyond the scope of this work, the logic behind the last Proposition seems to extend to this case as well. More precisely, our proof can be directly adapted to show that the seller with the highest cost (also the one with the highest reserve price) must post a reserve price that converges to his cost of production as the market becomes large. However, a convergence result for sellers with lower cost levels does not follow directly.

3 Symmetric pure strategy equilibrium

Let us address the question whether an equilibrium exist for a given f, n, k, a where sellers use pure strategies. Burguet and Sakovics (1999) established that when $a = 0$ and there are $k = 2$ sellers, such a pure strategy equilibrium does not exist. For reasons of tractability, we concentrate on *symmetric* pure strategy equilibria where the buyers employ symmetric strategies and the sellers play symmetric and pure strategies all posting reserve price $r \geq 0$. In what follows we assume that $r \leq a$ in equilibrium and thus all types are served.[4] In the online Appendix we establish the following result:

Theorem 3 *There is at most one symmetric pure strategy equilibrium of the game. In the symmetric pure strategy equilibrium each seller posts a reserve price*

$$r = \frac{(n-1)a}{(n-1) + (k-1)^2}. \tag{5}$$

A necessary condition for such an equilibrium to exist is that

$$af(a) \geq a^* = \frac{k - 1 + \frac{n-k}{k}}{k^2 - 2 + n - k}.$$

As Burguet and Sakovics (1999) have already pointed out, the first order conditions are not sufficient for seller optimization when $a = 0$. In that case the local second order condition fails and a symmetric pure strategy equilibrium does not exist. In the online Appendix we provide sufficient conditions for existence of a symmetric pure strategy equilibrium. Unfortunately, a simple condition stated on the primitives is not available, but we prove that if there are $n = 2$ buyers, then it is sufficient if F is convex, and we conjecture that convexity of F is sufficient for any n, k.

Let us provide an intuition for why the second order condition of seller optimization fails when $a = 0$, but may hold when a is large enough. When all sellers post the same reserve price r, then by unilaterally increasing his reserve price seller 1 will be visited with probability $1/n$ by all types above y. Therefore, one can think of y as being the choice variable of the seller. When seller 1 chooses r himself then $y = a$, but when he chooses a reserve strictly above r then $y > a$ and seller 1 loses the visitors with types between a and y. The key is to understand the costs and benefits of increasing y from a to $a + \varepsilon$, i.e. increasing the reserve price slightly above r. If seller 1 had two buyers visiting in the situation when $y = a$ and one of them had type less than $a + \varepsilon$, then by increasing y to $a + \varepsilon$ seller 1 loses a revenue of $a - r$ approximately. If seller 1 is visited by only one bidder, whose type is greater than $a + \varepsilon$, then seller 1 pockets a gain that is

[4] One can show that there is no symmetric pure strategy equilibrium where this condition fails.

equal to the increase in the reserve price. Finally, if seller 1 is visited by only one bidder, whose type is less than $a+\varepsilon$, then seller 1 loses a revenue of r. Note, that as a increases both r and $a-r$ increase and thus as a goes up it becomes more costly to turn off potential buyers by increasing r_1. On the other hand, when $a=r=0$ it is barely profitable to attract an extra buyer with valuation very close to a and thus increasing the reserve price is costless.

Theorem 3 implies that as the number of sellers (buyers) increases the sellers post lower (higher) equilibrium reserve prices to compete effectively. The equilibrium reserve price decreases in the size of the market due to the fact that each seller has less effect on the equilibrium utility levels of the buyers and thus increasing his reserve price becomes less appealing for each seller. It holds as before, that if the market grows large (and $\lim \frac{n-1}{(k-1)^2}=0$), then the equilibrium reserve price converges to zero.

4 Conclusion

Our contribution to the literature on competing auctions is twofold. First, we analyze the case of large markets and show that as the market becomes large, the utility effect of each single seller approaches zero. Therefore, as long as his reserve price is larger than his cost, it is strictly profitable for each seller to attract extra buyers by decreasing his reserve price. Hence in large markets, the sellers post reserve prices close to their production costs. We extend this result also to the case where a pure strategy equilibrium does not exist. The major contribution here lies in that we show that the reserve prices posted in large, but finite markets converge to production costs; an analysis that has not been provided before. Second, we provide conditions under which a pure strategy equilibrium exists and provide explicit solutions for the case of finite markets. This allows us to obtain intuitive comparative statics results, showing that if the number sellers (buyers) increases then the equilibrium reserve price goes down (up), and as the market size increases the equilibrium reserve price decreases.

5 Appendix

Lemma 4 *In case 1 it holds for all $x>t_{k-1}$ that*

$$0 \geq \frac{\partial u(x)}{\partial t_{k-1}} \geq -\frac{n-1}{k(k-1)} f(t_{k-1})(a+1).$$

Proof. Let $\Omega(x)$ denote the probability that type x is receiving an object. Since a type $x \in (t_{k-2}, t_{k-1})$ visits seller $k-1$, *any* of the other $n-1$ buyers takes the object away from a buyer with type x if this other buyer has a type between x and t_{k-1} and visits seller $k-1$ or has a type higher than t_{k-1} and visits seller $k-1$. (In what follows we will not explicitly use the superscript l, but all elements of strategic interaction k, n, t and the reserve prices depend on l.) Using Lemma 1, the first possibility occurs with probability $\frac{F(t_{k-1})-F(x)}{k-1}$, while the second occurs with probability $\frac{1-F(t_{k-1})}{k}$. Therefore, any given other buyer does *not* take the object away from a buyer with type x with probability

$$1-\frac{F(t_{k-1})-F(x)}{k-1}-\frac{1-F(t_{k-1})}{k}=\frac{k-1}{k}+\frac{F(x)}{k-1}-\frac{F(t_{k-1})}{k(k-1)}.$$

Therefore, considering the case where *none* of the other $n-1$ buyers take the object away from a buyer with type x one obtains

$$\Omega(x)=(\frac{k-1}{k}+\frac{F(x)}{k-1}-\frac{F(t_{k-1})}{k(k-1)})^{n-1}.$$

After differentiation we obtain that for all $x \in (t_{k-2}, t_{k-1})$

$$\frac{\partial \Omega(x)}{\partial t_{k-1}}=-\frac{n-1}{k(k-1)}(\frac{k-1}{k}+\frac{F(x)}{k-1}-\frac{F(t_{k-1})}{k(k-1)})^{n-2} f(t_{k-1}). \tag{6}$$

A similar argument implies that for all $x>t_{k-1}$

$$\Omega(x)=(1-\frac{1}{k}(1-F(x)))^{n-1}, \tag{7}$$

which does not depend on t_{k-1}.

Let $x \in (t_{k-2}, t_{k-1})$ and calculate the utility such a type achieves in equilibrium. The probability that no other buyer visits seller $k-1$ is $(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-1}$, in which case the buyer with type x pays r_{k-1} for the object, which he wins for sure. The probability that $g \in \{1, 2, ..., n-1\}$ other buyers visit seller $k-1$ and all visitors have types less than x is $\binom{n-1}{g}(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-1-g}(\frac{F(x)-F(t_{k-2})}{k-1})^g$. In this case the payment of a buyer with type x is equal to the largest valuation among all the $g-1$ other buyers who visit seller $k-1$. Therefore, his (expected) utility can be written as

$$u(x) = (\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-1}(x - r_{k-1}) +$$

$$+(n-1)(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-2}\frac{F(x)-F(t_{k-2})}{k-1}(x - E[y \mid y \in [t_{k-2}, x]]) +$$

$$+\binom{n-1}{2}(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-3}(\frac{F(x)-F(t_{k-2})}{k-1})^2(x - E[y^1 \mid y^1, y^2 \in [t_{k-2}, x], y^1 > y^2]) + ... \quad (8)$$

Now, take the decision problem of seller k in terms of choosing t_{k-1}, (which we can do since t_{k-1} and r_k are in a one-to-one relationship by Lemma 1) and let us calculate the utility change of a type when seller k decreases his decision variable t_{k-1} slightly. One needs to allow all the other cutpoints to change to accommodate the change in t_{k-1}, and thus when derivatives are taken with respect to t_{k-1} these indirect effects are also taken into account in what follows. Fixing x at the *initial* value of t_{k-2} we obtain that

$$\frac{\partial((\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-2}(F(x) - F(t_{k-2}))(x - E[y \mid y \in [t_{k-2}, x]]))}{\partial t_{k-1}} \mid_{x=t_{k-2}} = 0,$$

because $F(x) - F(t_{k-2}) = x - E[y \mid y \in [t_{k-2}, x]] = 0$. The derivatives of the other terms of $u(x)$ at $x = t_{k-2}$, except for the first one, are zero for the same reason and thus[5]

$$\frac{\partial u(x)}{\partial t_{k-1}} \mid_{x=t_{k-2}} = \frac{n-1}{(k-1)}(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-2}(t_{k-2} - r_{k-1})(\frac{\partial t_{k-2}}{\partial t_{k-1}} f(t_{k-2}) - \frac{f(t_{k-1})}{k}). \quad (9)$$

Now, by Lemma 1, it follows that $\frac{\partial u(x)}{\partial t_{k-1}} \mid_{x=t_{k-2}} \leq 0$ and $\frac{\partial t_{k-2}}{\partial t_{k-1}} \geq 0$, therefore

$$0 \leq \frac{n-1}{(k-1)}(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-2}(t_{k-2} - r_{k-1})\frac{\partial t_{k-2}}{\partial t_{k-1}} f(t_{k-2}) \leq$$

$$\leq \frac{n-1}{k(k-1)}(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-2}(t_{k-2} - r_{k-1}) f(t_{k-1}). \quad (10)$$

Then revisiting (9) yields that

$$0 \geq \frac{\partial u(x)}{\partial t_{k-1}} \mid_{x=t_{k-2}} \geq -\frac{n-1}{k(k-1)}(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-2}(t_{k-2} - r_{k-1}) f(t_{k-1}). \quad (11)$$

Using the envelope theorem we obtain that for all $x \geq z$

$$u(x) = u(z) + \int_z^x \Omega(y) dy$$

and thus

$$\frac{\partial u(x)}{\partial t_{k-1}} = \frac{\partial u(x)}{\partial t_{k-1}} \mid_{x=t_{k-2}} + \int_{t_{k-2}}^x \frac{\partial \Omega(y)}{\partial t_{k-1}} dy.$$

[5] When calculating the utility of type x (fixed at the initial level of t_{k-2}), we are using that when t_{k-1} decreases, then t_{k-2} decreases as well, so a type x that is equal to the initial value of t_{k-2} still visits seller $k-1$.

Formulas (6) and (7) imply that for all $x > t_{k-1}$ it holds that

$$\frac{\partial u(x)}{\partial t_{k-1}} = \frac{\partial u(x)}{\partial t_{k-1}} |_{x=t_{k-2}} - \frac{n-1}{k(k-1)} \int_{t_{k-2}}^{x} (\frac{k-1}{k} + \frac{F(y)}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-2} f(t_{k-1}) dy.$$

Then (11) implies that for all $x > t_{k-1}$ it holds that

$$0 \geq \frac{\partial u(x)}{\partial t_{k-1}} \geq$$

$$\geq -\frac{n-1}{k(k-1)} (\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-2} (t_{k-2} - r_{k-1}) f(t_{k-1}) -$$
$$-\frac{n-1}{k(k-1)} \int_{t_{k-2}}^{x} (\frac{k-1}{k} + \frac{F(y)}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-2} f(t_{k-1}) dy.$$

Using that $(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-2} < 1$ and $\int_{t_{k-2}}^{x} (\frac{k-1}{k} + \frac{F(y)}{k-1} - \frac{F(t_{k-1})}{k(k-1)})^{n-2} < 1$ and $x \leq a+1$ implies together with the above formula that indeed

$$0 \geq \frac{\partial u(x)}{\partial t_{k-1}} \geq -\frac{n-1}{k(k-1)} f(t_{k-1})(a+1).$$

■

Lemma 5 *If q sellers post the highest reserve price r_k, then if any single seller (seller k) of those q deviates in such a way t_{k-q} decreases below the original value, then it holds that*

$$0 \geq \int_{t^*_{k-q}}^{a+1} f(x) \frac{\partial u(x)}{\partial t_{k-q}} |_{t_{k-q}=t^*_{k-q}} \, dx \geq -\frac{\rho f(t_{k-q}) k}{(k-q)(q-1)}.$$

Proof of Lemma 5:

Proof. Note that

$$\frac{\partial u(a+1)}{\partial t_{k-q}} = \frac{\partial u(a+1)}{\partial t_{k-q+1}} \frac{\partial t_{k-q+1}}{\partial t_{k-q}} \tag{12}$$

and that since type $a+1$ visits the sellers who posted the highest reserve price and that reserve price does not change when seller k decreases r_k slightly, therefore $u(a+1)$ changes only because t_{k-q+1} changes. Let us show now that $\frac{\partial u(a+1)}{\partial t_{k-q+1}}$ is uniformly bounded. To see this, let $A(x)$ denote the probability that no other buyer with type above x visits seller $k-q+1$, the seller that the highest type, $a+1$ visits with positive probability. Then for all $x \geq t_{k-q+1}$

$$A(x) = (\frac{k-1}{k} + \frac{F(x)}{k})^{n-1}.$$

Then

$$u(a+1) = (\frac{k-1}{k} + \frac{F(t_{k-q+1})}{k})^{n-1}(a+1-r_{k-q+1}) + \int_{t_{k-q+1}}^{a+1} A'(x)(a+1-x) dx$$

and

$$\frac{\partial u(a+1)}{\partial t_{k-q+1}} = \frac{(n-1)}{k} (\frac{k-1}{k} + \frac{F(t_{k-q+1})}{k})^{n-1} f(t_{k-q+1})(t_{k-q+1} - r_{k-q+1}) \leq \rho f(t_{k-q+1}).$$

Therefore, the last equation and formula (12) imply that

$$-\frac{\rho f(t_{k-q}) k}{(k-q)(q-1)} \leq \frac{\partial u(a+1)}{\partial t_{k-q}} \leq 0. \tag{13}$$

Let $\Omega(x)$ denote the probability of winning for a buyer with type x. For $x \geq t_{k-q+1}$ it holds that

$$\Omega(x) = (1 - \frac{1 - F(x)}{k})^{n-1}.$$

Using the envelope formula,

$$u(x) = u(a+1) - \int_x^{a+1} \Omega(y) dy.$$

Therefore, for all $x \geq t_{k-q+1}$

$$\frac{\partial u(x)}{\partial t_{k-q}} = \frac{\partial u(a+1)}{\partial t_{k-q}}. \tag{14}$$

Using formulas (13) and (14) implies that indeed

$$0 \geq \int_{t^*_{k-q}}^{a+1} f(x) \frac{\partial u(x)}{\partial t_{k-q}} |_{t_{k-q}=t^*_{k-q}} dx = \int_{t^*_{k-q}}^{a+1} f(x) \frac{\partial u(a+1)}{\partial t_{k-q}} |_{t_{k-q}=t^*_{k-q}} dx =$$

$$= (1 - F(t^*_{k-q})) \frac{\partial u(a+1)}{\partial t_{k-q}} \geq \frac{\partial u(a+1)}{\partial t_{k-q}} \geq -\frac{\rho f(t_{k-q}) k}{(k-q)(q-1)}.$$

■

Proof for Case 2:

Proof. In Case 2 the main difference is that with positive probability there may be $q > 1$ sellers who post the highest reserve prices. The proof below establishes exactly that if there are $q > 1$ sellers who post the highest reserve price, then any of those q sellers find it profitable to decrease his reserve price. First, take the case where $q = k$, i.e. all the other sellers post reserve price r_k as well. As we show it in the online Appendix in the proof of Theorem 3, seller k has an incentive to decrease his reserve price if $r_k > \frac{a(n-1)}{(n-1)+(k-1)^2}$. But this threshold is approaching zero and thus if the market is large enough, seller k has an incentive to decrease his price for any positive r_k.

Otherwise (if $q < k$), let $r_1 \leq r_2 \leq ... \leq r_{k-q-1} < r_{k-q} < r_{k-q+1} = ... = r_k < a+1$, and suppose that seller k decreases r_k slightly. In this case Lemma 1 implies that after this change seller k is visited with probability $\frac{1}{k-q+1}$ by types between t_{k-q} and t_{k-q+1} and with probability $\frac{1}{k}$ by types larger than t_{k-q+1}. When r_k is at the original level (and thus $r_{k-q+1} = ... = r_k$), then of course $t_{k-q} = t_{k-q+1}$, but when r_k is decreased then $t_{k-q} < t_{k-q+1}$. The following useful result (similar to Lemma 1 in content) helps the analysis below[6]:

Lemma 6 *The cutoff values are differentiable in r_k from the left hand side and it holds that $\frac{\partial t_{k-q}}{\partial r_k} > 0$, $\frac{\partial t_{k-q-1}}{\partial r_k} > 0$ and $\frac{\partial t_{k-q+1}}{\partial r_k} < 0$. Moreover, for all $x \geq t_{k-q-1}$ it holds that $\frac{\partial u(x)}{r_k} < 0$.*

Because of the above Lemma, instead of r_k one can take t_{k-q} as the choice variable of seller k. Take a buyer with type x that is equal to the original value of t_{k-q-1}. For that type it is optimal to visit seller $k-q$. Suppose that t_{k-q} goes down and thus t_{k-q-1} goes down as well by the Lemma 6. Then for type x it is still optimal to visit seller $k-q$. Lemma 5 implies that we can bound the utility effect of such a change as:

$$0 \geq \int_{t^*_{k-q}}^{a+1} f(x) \frac{\partial u(x)}{\partial t_{k-q}} |_{t_{k-q}=t^*_{k-q}} dx \geq -\frac{\rho f(t_{k-q}) k}{(k-q)(q-1)}. \tag{15}$$

We first provide a uniform convergence result for the case where q is not too small. For all $x \geq t_{k-q-1}$ let $G(x)$ denote the probability that seller k sells to a buyer with type less than x or does not sell at all and let $g(x) = \frac{\partial G}{\partial x}$ denote the corresponding density function. If $x > t_{k-q+1}$, this event happens if and only if no buyer with type greater than x visits seller k, and thus $G(x) = (1 - \frac{1-F(x)}{k})^n$. For $x \in [t_{k-q}, t_{k-q+1}]$ the following holds:

$$G(x) = (1 - \frac{1 - F(t_{k-q+1})}{k} - \frac{F(t_{k-q+1}) - F(x)}{k-q+1})^n.$$

[6] The proof of this Lemma is very similar to that of Lemma 1 and is thus omitted.

The expected revenue of seller k as the total surplus generated at seller k minus the total utilities of the types visiting seller k or

$$R_k = W_k - C_k$$

with

$$W_k = \int_{t_{k-q}}^{a+1} xg(x)dx$$

and

$$C_k = n\int_{t_{k-q}}^{t_{k-q+1}} \frac{1}{k-q+1}u(x)f(x)dx + n\int_{t_{k-q+1}}^{a+1} \frac{1}{k}u(x)f(x)dx,$$

since types above t_{k-q+1} visit seller k with probability $1/k$ and types in $[t_{k-q}, t_{k-q+1}]$ visit with probability $\frac{1}{k-q+1}$. It is useful to describe the utility cost in an alternative way using the function u^* that describes the utility of a type conditional on obtaining the object from seller k:

$$C_k = \int_{t_{k-q}}^{a+1} u^*(x)g(x)dx.$$

Note that type t_{k-q} obtains the object from seller k if and only if no other buyer visited seller k and thus his utility conditional on obtaining the object is $u^*(t_{k-q}) = t_{k-q} - r_k$. With this formulation (and explicitly recognizing the cutpoints) one can rewrite the revenue as

$$W_k = \int_{t_{k-q}}^{t_{k-q+1}} (x - u^*(x))\frac{n}{k-q+1}(1 - \frac{1-F(t_{k-q+1})}{k} - \frac{F(t_{k-q+1}) - F(x)}{k-q+1})^{n-1}f(x)dx +$$

$$+\int_{t_{k-q+1}}^{a+1} (x - u^*(x))\frac{n}{k}(1 - \frac{1-F(x)}{k})^{n-1}f(x)dx.$$

Using the above definitions implies that

$$\frac{\partial R_k}{\partial t_{k-q}} = -\frac{n}{k-q+1}(1 - \frac{1-F(t_{k-q})}{k})^{n-1}f(t_{k-q})r_k - \int_{t_{k-q}}^{t_{k-q+1}} (x - u^*(x))\frac{\partial g(x)}{\partial t_{k-q+1}}\frac{\partial t_{k-q+1}}{\partial t_{k-q}}dx$$

$$+\frac{\partial t_{k-q+1}}{\partial t_{k-q}}f(t_{k-q+1})(\frac{1}{k-q+1} - \frac{1}{k}) - \frac{n}{k}\int_{t_{k-q+1}}^{a+1} f(x)\frac{\partial u(x)}{\partial t_{k-q}}dx - \frac{n}{k-q+1}\int_{t_{k-q}}^{t_{k-q+1}} f(x)\frac{\partial u(x)}{\partial t_{k-q}}dx.$$

We need to evaluate this derivative at the point where $r_k = r_{k-1} = ... = r_{k-q+1}$ and thus $t_{k-q} = t_{k-q+1} = t^*_{k-q}$, where t^*_{k-q} stands for the original cutpoint. Therefore,

$$\frac{\partial R_k}{\partial t_{k-q}} |_{t_{k-q}=t^*_{k-q}} = -\frac{n}{k-q+1}(1 - \frac{1-F(t^*_{k-q})}{k})^{n-1}f(t^*_{k-q})r_k +$$

$$+\frac{\partial t_{k-q+1}}{\partial t_{k-q}} |_{t_{k-q}=t^*_{k-q}} f(t^*_{k-q+1})(\frac{1}{k-q+1} - \frac{1}{k}) - \frac{n}{k}\int_{t^*_{k-q}}^{a+1} f(x)\frac{\partial u(x)}{\partial t_{k-q}} |_{t_{k-q}=t^*_{k-q}} dx.$$

Using that $q > 1$ and $\frac{\partial t_{k-q+1}}{\partial t_{k-q}} \leq 0$ by Lemma 6 and formula (15) yields that

$$\frac{\partial R_k}{\partial t_{k-q}} |_{t_{k-q}=t^*_{k-q}} \leq -\frac{n}{k-q+1}(1 - \frac{1-F(t^*_{k-q})}{k})^{n-1}f(t^*_{k-q})r_k - \frac{n}{k}\int_{t^*_{k-q}}^{a+1} f(x)\frac{\partial u(x)}{\partial t_{k-q}} |_{t_{k-q}=t^*_{k-q}} dx \leq$$

$$\leq -\frac{n}{k-q+1}(1 - \frac{1-F(t^*_{k-q})}{k})^{n-1}f(t^*_{k-q})r_k + \frac{\rho f(t^*_{k-q})n}{(k-q)(q-1)} < 0$$

if

$$\frac{k-q+1}{(k-q)(q-1)} < (1 - \frac{1-F(t^*_{k-q})}{k})^{n-1}\frac{r_k}{\rho}.$$

On the other hand if $k \geq 2$

$$(1-\frac{1-F(t^*_{k-q})}{k})^{n-1} \geq (1-\frac{1}{k})^n \geq (1-\frac{1}{k})^{k\rho} \geq (\frac{1}{4})^\rho.$$

Therefore, it is sufficient to show that

$$\frac{k-q+1}{(k-q)(q-1)} < (\frac{1}{4})^\rho \frac{r_k}{\rho} = T.$$

Let q^* solve $\frac{2}{q^*-1} = T$ and let $q \geq q^*+1$. Then since $k-q \geq 1$ it follows that

$$\frac{k-q+1}{(k-q)(q-1)} \leq \frac{2}{q-1} < T$$

and thus for any value of $k \geq 2$ and any reserve prices posted by the other $k-q$ sellers who did not post r_k, if $q \geq q^*+1$, then seller k has an incentive to reduce his reserve price.

Next consider the case where $\lim_{l\to\infty} q \leq q^* < \infty$. Note, that in this case it is sufficient to show that the left hand derivative of R_k (when r_k is maximal) is negative for all $q \leq q^*$ if $k > \underline{k}(q)$, because then letting $\underline{k} = \max\{\underline{k}(1), \underline{k}(2, ..., \underline{k}(q^*)\}$ may serve as a uniform bound, so that convergence is uniform in q.[7] Let H denote the probability that a given other buyer does not visit seller $k-q$. Then

$$H = 1 - \frac{F(t_{k-q}) - F(t_{k-q-1})}{k-q} - \frac{F(t_{k-q+1}) - F(t_{k-q})}{k-q+1} - \frac{1-F(t_{k-q+1})}{k} \tag{16}$$

by using Lemma 1. A similar argument as before (8) implies that

$$u(x) = H^{n-1}(t_{k-q-1} - r_{k-q}) + (n-1)\frac{F(x) - F(t_{k-q-1})}{k-q} H^{n-2} E[y \mid y \in [t_{k-q-1}, x]] + ...$$

Similar argument as after equation (8) implies that all the terms except for the first one are higher order in t_{k-q-1} when x is close to t_{k-q-1}. Therefore,

$$\frac{\partial u(x)}{\partial t_{k-q}} |_{x=t_{k-q-1}} = (n-1)H^{n-2}(t_{k-q-1} - r_{k-q})\frac{dH}{dt_{k-q}}, \tag{17}$$

where $\frac{dH}{dt_{k-q}}$ stands for the derivative of H with respect to t_{k-q} taking indirect effects through t_{k-q-1} and t_{k-q+1} into account.

Using (17) and Lemma 6 implies that $\frac{\partial H}{\partial t_{k-q}} \leq 0$. This implies that

$$\frac{f(t_{k-q})}{(k-q)(k-q+1)} \geq -\frac{f(t_{k-q+1})(q-1)}{k(k-q+1)}\frac{\partial t_{k-q+1}}{\partial t_{k-q}} + \frac{f(t_{k-q-1})}{(k-q)}\frac{\partial t_{k-q-1}}{\partial t_{k-q}}. \tag{18}$$

Lemma 6 also implies that $\frac{\partial t_{k-q-1}}{\partial t_{k-q}} \geq 0$ and thus it follows from the last inequality that

$$\frac{f(t_{k-q})}{(k-q)(k-q+1)} \geq -\frac{f(t_{k-q+1})(q-1)}{k(k-q+1)}\frac{\partial t_{k-q+1}}{\partial t_{k-q}}$$

or

$$\frac{f(t_{k-q})k}{f(t_{k-q+1})(k-q)(q-1)} \geq -\frac{\partial t_{k-q+1}}{\partial t_{k-q}}. \tag{19}$$

Also, by Lemma 6 it holds that $0 \geq \frac{\partial t_{k-q+1}}{\partial t_{k-q}}$ and thus

$$0 \geq \frac{\partial t_{k-q+1}}{\partial t_{k-q}} \geq -\frac{f(t_{k-q})k}{f(t_{k-q+1})(k-q)(q-1)}. \tag{20}$$

[7] Convergence is uniform in the reserve prices posted by the other sellers just like in Case 1.

Equation (19) implies that $\frac{\partial t_{k-q+1}}{\partial t_{k-q}}$ is bounded in absolute value. Formula (18) implies together with Lemma 6 that

$$\frac{f(t_{k-q})}{k-q+1} \geq f(t_{k-q-1})\frac{\partial t_{k-q-1}}{\partial t_{k-q}}$$

and thus

$$\lim_{l\to\infty}\frac{\partial t_{k-q-1}}{\partial t_{k-q}} = 0. \tag{21}$$

Now, we establish that $\lim \frac{\partial u(x)}{\partial t_{k-q}} \mid_{x=t_{k-q-1}}= 0$, which is equivalent to (see (17)) $\lim(n-1)\frac{dH}{dt_{k-q}} = 0$. By definition

$$(n-1)\frac{dH}{dt_{k-q}} = (n-1)\frac{\partial H}{\partial t_{k-q}} + (n-1)\frac{\partial H}{\partial t_{k-q+1}}\frac{\partial t_{k-q+1}}{\partial t_{k-q}} + (n-1)\frac{\partial H}{\partial t_{k-q-1}}\frac{\partial t_{k-q-1}}{\partial t_{k-q}}.$$

Using formulas (16), (20), (21) together with $q \leq q^*$ implies that indeed $\lim(n-1)\frac{dH}{dt_{k-q}} = 0$ and thus $\lim \frac{\partial u(x)}{\partial t_{k-q}} \mid_{x=t_{k-q-1}}= 0$. The envelope theorem implies that for all $x > t_{k-q-1}$

$$u(x) = u(t_{k-q-1}) + \int_{t_{k-q-1}}^{x} \Omega(y)dy,$$

where $\Omega(y)$ denotes the winning probability of type y. It is straightforward to establish that $\lim \frac{\partial \Omega(y)}{\partial t_{k-q}} = 0$ for all $y > t_{k-q-1}$, and thus it holds that for all $x > t_{k-q}$

$$\lim \frac{\partial u(x)}{\partial t_{k-q}} = 0.$$

To complete the proof it is sufficient to show that $\lim \frac{\partial W_k}{\partial t_{k-q}} > 0$, which can be done following similar arguments as in the other cases. ■

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Online Appendix

Proof of Lemma 1:

Proof. Proving that any equilibria can be characterized by such cutoff values is a straightforward extension of the proof for the case of two sellers as covered in Burguet and Sakovics (1999). Now, we turn to proving uniqueness of those values. For notational simplicity we assume that all the reserve prices are different, but the proof can be easily extended to the case when some of the reserve prices are equal to each other. We restrict attention to the case where all k sellers are visited with positive probability by some buyer types, otherwise seller k (the seller with the highest reserve price) could obviously increase his profits by lowering his reserve price until he is visited with positive probability.[8] Fix a utility level for the highest type and denote it by $\overline{u} = u(a+1)$. The utility of the the highest type is equal to his utility from showing up at seller k . At that seller only other buyers with types above t_{k-1} show up and they do it with probability $1/k$. With probability $F(t_{k-1}) + \frac{k-1}{k}(1-F(t_{k-1}))$ no other buyer shows up, in which case the utility of type $a+1$ is equal to $a+1-r_k$. Otherwise, if z is the highest type of another buyer who visited seller k then the utility of type $a+1$ is $a+1-z$. Let $B(z)$ denote the probability that either no other buyer visits seller k or all the buyers visiting have types less than z. Then by construction

$$B(z) = (F(t_{k-1}) + \frac{k-1}{k}(1-F(t_{k-1})) + \frac{F(z)-F(t_{k-1})}{k})^{n-1}.$$

Therefore, it holds that

$$\overline{u} = (a+1) - (F(t_{k-1}) + \frac{k-1}{k}(1-F(t_{k-1})))r_k - \int_{t_{k-1}}^{a+1} B'(z)zdz. \tag{22}$$

One can easily see that the right hand side of the above equation is increasing in t_{k-1} and thus for a given $\overline{u}$ there is a unique solution for t_{k-1}. Take this solution for t_{k-1} and note that

$$u(t_{k-1}) = (F(t_{k-1}) + \frac{k-1}{k}(1-F(t_{k-1})))^{n-1}(t_{k-1} - r_k),$$

since such a type wins the object at seller k if and only if no other buyer has visited seller k. One can also calculate the utility of type t_{k-1} from visiting seller $k-1$, since that type is indifferent between visiting sellers $k-1$ and k. Equating the utilities of type t_{k-1} from visiting buyer $k-1$ and buyer k provides us with an equation in t_{k-2} and t_{k-1} only. However, t_{k-1} was already pinned down by $\overline{u}$ and thus now t_{k-2} can be pinned down as well. An iterative argument pins down the values t_1 through t_{k-3} as well. Therefore, to conclude that the cutoff types are unique it is sufficient to show that $\overline{u}$ is uniquely determined by the reserve prices posted.

From (22) and the fact that its right hand side is increasing in t_{k-1}it follows that

$$\frac{\partial t_{k-1}}{\partial \overline{u}} > 0. \tag{23}$$

Suppose now that $\frac{\partial t_{k-2}}{\partial \overline{u}} < 0$ and we arrive at a contradiction. More precisely, let $\widetilde{\overline{u}} > \overline{u}$ describe two candidate utility values for the highest type and suppose that $\widetilde{t}_{k-2} < t_{k-2}$. Then note that (for a small difference between $\widetilde{\overline{u}}$ and $\overline{u}$) $\widetilde{t}_{k-1} > t_{k-1} > t_{k-2} > \widetilde{t}_{k-2}$. Let $\Omega(x)$ and $\widetilde{\Omega}(x)$ be the probability that a buyer with type x obtains the object when $u(a+1) = \widetilde{\overline{u}}$ or $\overline{u}$. Since by construction $\widetilde{t}_{k-1} > t_{k-1}$, therefore for all the types $x > \widetilde{t}_{k-1}$ it holds that

$$\Omega(x) = \widetilde{\Omega}(x) = (F(x) + \frac{k-1}{k}(1-F(x)))^{n-1}. \tag{24}$$

Now, for the same reason as before it holds for all $x \in (t_{k-1}, \widetilde{t}_{k-1})$ that

$$\Omega(x) = (F(x) + \frac{k-1}{k}(1-F(x)))^{n-1}.$$

[8] One can establish (using techniques similar to the ones below) that there cannot both be an equilibrium where seller k is not visited by any buyer type and an equilibrium in which he is visited by some buyer types.

However, for all $x \in (t_{k-1}, \widetilde{t}_{k-1})$

$$\Omega(x) = (F(x) + \frac{k-2}{k-1}(F(\widetilde{t}_{k-1}) - F(x)) + \frac{k-1}{k}(1 - F(\widetilde{t}_{k-1})))^{n-1},$$

because the probability that another buyer with type between x and $\widetilde{t}_{k-1}$ visits the same seller as type x (seller $k-1$) is $1/(k-1)$. A simple comparison yields that for all $x \in (t_{k-1}, \widetilde{t}_{k-1})$

$$\Omega(x) > \widetilde{\Omega}(x). \tag{25}$$

The envelope theorem implies that

$$u'(x) = \Omega(x) \tag{26}$$

and

$$\widetilde{u}'(x) = \widetilde{\Omega}(x). \tag{27}$$

Using (24), (26), (27) and (25) imply that

$$\widetilde{u}(t_{k-1}) > u(t_{k-1}). \tag{28}$$

Now, let us compare the utility of type t_{k-1} in the two regimes in a more direct way. This type visits seller $k-1$ no matter which utility value we use for the highest type, because $\widetilde{t}_{k-1} > t_{k-1} > \widetilde{t}_{k-2}$. Also, seller $k-1$ is visited with more buyer types in the regime when the utility of the highest type is $\widetilde{\overline{u}}$ than when the utility is $\overline{u}$. Therefore, type $k-1$ faces more competition in the $\widetilde{\overline{u}}$ than in the $\overline{u}$ regime with the reserve price r_{k-1} being the same and thus

$$\widetilde{u}(t_{k-1}) < u(t_{k-1}).$$

But this last formula contradicts (28) and thus $\widetilde{t}_{k-2} > t_{k-2}$ holds. Extending this argument iteratively we obtain for all j that

$$\frac{\partial t_j}{\partial \overline{u}} > 0. \tag{29}$$

The above iterative construction also implies that for all $x \in (r_1, a+1)$ it holds that

$$\frac{\partial u(x)}{\partial \overline{u}} > 0. \tag{30}$$

Now, take a type $y \in (r_1, t_1)$ that visits seller 1 according to the cutpoints received from our iterative construct. For such a type it is the case that the higher all the cutpoints are the more likely that seller 1 is visited by the other buyers and thus his utility is lower. Therefore, (29) implies that

$$\frac{\partial u(y)}{\partial \overline{u}} < 0. \tag{31}$$

But then there cannot be two different initial values $\overline{u}$, because that would imply (via (30)) that $u(y)$ is higher for higher values of $\overline{u}$, but (31) would imply the opposite movement which cannot both hold. Therefore, only a unique value of $\overline{u}$ exists, which concludes the proof of uniqueness. If r_k is strictly greater than the other reserve prices, then differentiability of the cutoff types (in r_k) directly follows from the implicit function theorem.

Finally, we prove the comparative statics results for the case where $r_k > r_{k-1}$. Take two situations where the first $k-1$ sellers have their reserve prices fixed and seller k considers choosing between r_k and $\widetilde{r}_k$. Variables without tilde refer to variables in the case where seller k posts r_k, while variables with tilde refer to the situations where seller k posts $\widetilde{r}_k$. Supposing that $\widetilde{r}_k > r_k$, and $t_{k-1} > \widetilde{t}_{k-1}$ hold at the same time we establish a contradiction below. If seller k posts r_k, then there are less buyers visiting seller k and the reserve price is also lower compared to the situation where seller k posts $\widetilde{r}_k$. Therefore, it must hold that

$$u(a+1) > \widetilde{u}(a+1). \tag{32}$$

Let $\Omega(x)$ and $\widetilde{\Omega}(x)$ be the probability that a buyer with type x obtains the object when the reserve price is r_k or $\widetilde{r}_k$.[9] The envelope theorem implies that

$$u'(x) = \Omega(x) \tag{33}$$

and

$$\widetilde{u}'(x) = \widetilde{\Omega}(x). \tag{34}$$

Since by assumption $t_{k-1} > \widetilde{t}_{k-1}$ therefore for all the types $x > t_{k-1}$ it holds that

$$\Omega(x) = \widetilde{\Omega}(x) = (F(x) + \frac{k-1}{k}(1 - F(x)))^{n-1},$$

since type x loses if and only if there is a higher type visiting seller k, which occurs with the same probability in the two cases. Therefore, the last four formulas imply that

$$u(t_{k-1}) > \widetilde{u}(t_{k-1}). \tag{35}$$

Now, for the same reason as before it holds for all $x \in (\widetilde{t}_{k-1}, t_{k-1})$ that

$$\widetilde{\Omega}(x) = (F(x) + \frac{k-1}{k}(1 - F(x)))^{n-1}.$$

However, for all $x \in (\widetilde{t}_{k-1}, t_{k-1})$

$$\Omega(x) = (F(x) + \frac{k-2}{k-1}(F(t_{k-1}) - F(x)) + \frac{k-1}{k}(1 - F(t_{k-1})))^{n-1},$$

because the probability that another buyer with type between x and t_{k-1} visits the same seller as type x (seller $k-1$) is $1/(k-1)$. A simple comparison yields that $\widetilde{\Omega}(x) > \Omega(x)$ and then (33), (34) and (35) imply

$$u(\widetilde{t}_{k-1}) > \widetilde{u}(\widetilde{t}_{k-1}). \tag{36}$$

Now, suppose that $\widetilde{t}_{k-2} > t_{k-2}$. Then a similar argument as above implies that for all $x \in (\widetilde{t}_{k-2}, \widetilde{t}_{k-1})$ it holds that $\widetilde{\Omega}(x) > \Omega(x)$ and thus $\widetilde{u}'(x) > u'(x)$. Then (36) implies that

$$u(\widetilde{t}_{k-2}) > \widetilde{u}(\widetilde{t}_{k-2}). \tag{37}$$

By construction it is optimal for type $\widetilde{t}_{k-2}$ to visit seller $k-1$ when seller k posts a reserve price of $\widetilde{r}_k$. Also, since $t_{k-1} > \widetilde{t}_{k-1} > \widetilde{t}_{k-2} > t_{k-2}$ it follows that it is optimal for type $\widetilde{t}_{k-2}$ to visit seller $k-1$ when seller k posts a reserve price of r_k. Let us compare $u(\widetilde{t}_{k-2})$ and $\widetilde{u}(\widetilde{t}_{k-2})$ directly. Since $\widetilde{t}_{k-2} > t_{k-2}$ and $t_{k-1} > \widetilde{t}_{k-1}$, it follows that seller $k-1$ is visited with a strictly lower probability when k posted $\widetilde{r}_k$ then when he posted r_k and simple calculations show that it must hold that

$$u(\widetilde{t}_{k-2}) < \widetilde{u}(\widetilde{t}_{k-2}),$$

since the reserve price posted by seller $k-1$ is unchanged.

But the last inequality contradicts with (37) and thus $t_{k-2} > \widetilde{t}_{k-2}$ follows. Similar arguments as before formula (36) then imply that

$$u(\widetilde{t}_{k-2}) > \widetilde{u}(\widetilde{t}_{k-2}).$$

Proceeding iteratively one can show that for all j it holds that for all $j = 1, 2, ..., k-1$

$$t_j > \widetilde{t}_j \tag{38}$$

and also that for all x

$$u(x) > \widetilde{u}(x). \tag{39}$$

[9] This probability is the same regardless of which seller a buyer with type x visits among his optimal choices.

However, take a low type $y \in (r_1, \widetilde{t}_1)$ who visits seller 1 both when seller k posts r_k or when he posts $\widetilde{r}_k$. It is easy to see that if the cutoff values decrease then he is better off, because seller 1 is visited with a lower probability. Therefore, (38) implies that $\widetilde{u}(y) > u(y)$, which contradicts (39) establishing that $\frac{\partial t_{k-1}}{\partial r_k} > 0$. The result that $\frac{\partial t_{k-2}}{\partial r_k} > 0$ can be established by a similar iterative argument, which we omit. The claim that $u(x)$ decreases in r_k for all $x \geq t_{k-2}$ follows also from the envelope theorem and the iterative construction employed. ■

Proof of Theorem 3:

Proof. Suppose that all other sellers post reserve price r and seller 1 deviates to r_1. First, take the case where $r_1 > r$. Following Burguet and Sakovics, we can prove that the unique symmetric Bayesian equilibrium in the buyers' stage game is such that types less than y visit sellers 2 through k with equal probability and types above y visit all k sellers with equal probability. Type y has the same probability of winning at any seller, this probability is $(1 - \frac{1-F(y)}{k})$. At seller 1 type y wins if and only if all the other buyers visited other sellers and his payment is r_1 conditional on winning. Therefore, the expected payment of type y at seller 1 is $(1 - \frac{1-F(y)}{k})r_1$. If type y visits seller $j \neq 1$, then he wins if and only all the other buyers have visited a seller other than j or those visiting j have types less than y. If no one else visited j, then buyer i pays r and if l other buyers visited j, then the expected payment is the expected value of the highest type among the l buyers conditional on they all having types less than y (otherwise buyer i does not win). Let $Em^{(l)}$ denote this expected value when l other buyers visited.

Since his probability of winning is the same whether he visited seller 1 or $j \neq 1$, the payment of type y must be equal at seller 1 and at the other sellers. Therefore,

$$(1 - \frac{1-F(y)}{k})^{n-1} r_1 = (1 - \frac{1-F(y)}{k} - \frac{F(y)}{k-1})^{n-1} r + \tag{40}$$

$$+ \sum_{l=2}^{n} \binom{n-1}{l-1} (1 - \frac{(1-F(y)}{k} - \frac{F(y)}{k-1})^{n-l} (\frac{F(y)}{k-1})^{l-1} Em^{(l-1)}.$$

Let us rewrite expression $\varphi = \sum_{l=2}^{n} \binom{n-1}{l-1} (1 - \frac{(1-F(y)}{k} - \frac{F(y)}{k-1})^{n-l} (\frac{F(y)}{k-1})^{l-1} Em^{(l-1)}$ as an expected value of a random variable. Let τ take value zero when there is no other buyer at seller $j \neq 1$, and otherwise let τ take the highest type among the other buyers who visited j. Let function $\rho(z)$ be equal to z if $z \leq y$ and let $\rho(z) = 0$ if $z > y$. Then φ is the expected value of random variable $\rho(\tau)$. Letting t be the density function of τ, it holds that

$$\varphi = \int_a^{a+1} \rho(z) t(z) dz = \int_a^y z t(z) dz.$$

Also, by construction for all $z \leq y$ it holds that $t(z) = \frac{n-1}{k-1}(1 - \frac{1}{k} - \frac{F(y)}{k(k-1)} + \frac{F(z)}{k-1})^{n-2} f(z)$ and thus

$$\varphi = \int_a^y \frac{n-1}{k-1} z (1 - \frac{1}{k} - \frac{F(y)}{k(k-1)} + \frac{F(z)}{k-1})^{n-2} f(z) dz. \tag{41}$$

Now, we describe the expected revenue of seller 1. The revenue of seller 1 is equal to r_1 if exactly one buyer visited him, and zero if no buyer did. If more than two buyers visited him, then the revenue is equal to the value of the second highest type. Therefore, the expected revenue from having $l \geq 2$ buyers visiting is equal to $Es^{(l)}$, the expected value of the second highest type conditional on all buyers having types above y (otherwise they would have visited other sellers). With this shorthand notation the expected revenue of seller 1 is

$$R_1 = n \frac{1-F(y)}{k} (1 - \frac{1-F(y)}{k})^{n-1} r_1 +$$

$$+ \sum_{h=2}^{n} \binom{n}{h} (\frac{(1-F(y)}{k})^h (1 - \frac{1-F(y)}{k})^{n-h} Es^{(h)}. \tag{42}$$

A similar reasoning as before formula (41) yields that

$$\sum_{h=2}^{n} \binom{n}{h} (\frac{(1-F(y)}{k})^h (1 - \frac{1-F(y)}{k})^{n-h} Es^{(h)} = \tag{43}$$

$$= \frac{n}{k}\int_y^{a+1} z(n-1)(1-\frac{1-F(z)}{k})^{n-2}\frac{1-F(z)}{k}f(z)dz.$$

Equations (40), (41), (42) and (43) yield that

$$R_1 = \overline{R}_1(y) = \frac{n}{k}[\int_y^{a+1} z(n-1)(1-\frac{1-F(z)}{k})^{n-2}\frac{1-F(z)}{k}f(z)dz+$$

$$+(1-F(y))\{(1-\frac{1}{k}-\frac{F(y)}{k(k-1)})^{n-1}r+\int_a^y \frac{n-1}{k-1}z(1-\frac{1}{k}-\frac{F(y)}{k(k-1)}+\frac{F(z)}{k-1})^{n-2}f(z)dz\}]. \quad (44)$$

Now, let $r_1 < r$. In this case the unique symmetric Bayesian equilibrium in the buyers' stage game is such that types less than x visit seller 1 and types above x visit all k sellers with equal probability. For all $z \leq x$ let

$$G(z) = \frac{(1-F(x))(k-1)}{k} + F(z),$$

and for all $z > x$ let

$$G(z) = \frac{(k-1)+F(z)}{k}.$$

A similar analysis as above yields that inducing $x \in [a, a+1]$ yields an expected revenue of

$$R_1 = \widetilde{R}_1(x) = \quad (45)$$

$$\int_a^x zn(n-1)G(z)^{n-2}(1-G(z))f(z)dz + \int_x^{a+1} \frac{zn(n-1)G(z)^{n-2}(1-G(z))f(z)}{k}dz$$

$$+n\frac{(1+(k-1)F(x))}{k}[(1-\frac{1-F(x)}{k})^{n-1}r - \int_a^x z(n-1)f(z)G(z)^{n-2}dz].$$

The first order condition at $y = a$ is $\frac{\partial \overline{R}_1}{\partial y} |_{y=a} \leq 0$, which is equivalent to $r \geq \frac{(n-1)a}{(n-1)+(k-1)^2}$. The first order condition at $x = a$ is $\frac{\partial \widetilde{R}_1}{\partial a} |_{x=a} \leq 0$, which is equivalent to $r \leq \frac{(n-1)a}{(n-1)+(k-1)^2}$. Hence, the first order condition(s) uniquely pin down the reserve price at $r = \frac{(n-1)a}{(n-1)+(k-1)^2}$. For the local second order conditions we need that when we let $r = \frac{(n-1)a}{(n-1)+(k-1)^2}$, the second derivatives of $\overline{R}_1$ and $\widetilde{R}_1$ are non-positive at $y = a$ and $x = a$ respectively. The second derivative of $\overline{R}_1$ at $y = a$ is non-positive if $af(a) \geq a^*$, while that of $\widetilde{R}_1$ at $x = a$ is non-positive for all values of a and $f(a)$. ■

The following Corollary provides a sufficient condition for existence of a symmetric pure strategy equilibrium and states some specific cases in which those conditions hold:

Corollary 7 *A symmetric pure strategy equilibrium exists if functions $\overline{R}_1$ and $\widetilde{R}_1$ are quasiconcave for all $y \in [a, a+1]$ and $x \in [a, a+1]$ when $r = \frac{(n-1)a}{(n-1)+(k-1)^2}$. If the necessary conditions stated in Theorem 1 hold and $n = 2$ and F is convex, then there exists a symmetric pure strategy equilibrium.*

Proof. The first part of the Theorem is a simple consequence of the above analysis. For the $n = 2$ case a simple substitution yields that $\frac{\partial^2 \overline{R}_1}{\partial y^2}$ and $\frac{\partial^2 \widetilde{R}_1}{\partial x^2}$ are decreasing functions for all $y \in [a, a+1]$ and $x \in [a, a+1]$ when $r = \frac{(n-1)a}{(n-1)+(k-1)^2}$ and F is convex. Therefore, the local second order conditions at $y = a$ and $x = a$ are sufficient globally. ■

Unfortunately, a simple condition stated on the primitives is not available but we prove that if $n = 2$ then it is sufficient if F is convex, and we conjecture that convexity of F is sufficient for any n, k. The intuition of this result can be explained as follows: As Burguet and Sakovics (1999) have already pointed out, the first order conditions are not sufficient for seller optimization. Indeed, when $a = 0$ (the case studied in their paper), the local second order condition fails and a symmetric pure strategy equilibrium does not exist. In our model with a large enough, this problem does not arise, because with a high enough level of a it holds that $\frac{\partial^2 \overline{R}_1}{\partial y^2} |_{y=a} \leq 0$. In less technical terms, the key is to understand the costs and benefits of increasing y from a to $a + \varepsilon$. If seller 1 had two buyers visiting in the situation when $y = a$ and one of them had type less than $a + \varepsilon$, then by increasing y to $a + \varepsilon$ seller 1 loses a revenue of $a - r$ approximately. If seller 1 is visited by only one bidder, whose type is greater than $a + \varepsilon$, then seller 1 pockets a gain that is equal to the increase in the reserve price. Finally, if seller 1 is visited by only one bidder, whose type is less than $a + \varepsilon$, then seller 1 loses a revenue of r. Note, that as a increases both r and $a - r$ increase and thus as a goes up it becomes more costly to turn off potential buyers by increasing r_1. On the other hand, when $a = r = 0$ it is barely profitable to attract an extra buyer with valuation very close to a and thus increasing the reserve price is costless.

As the market becomes large it holds that $a^* \to 0$, and thus for any (fixed) positive values of a and $f(a)$, the local second order condition $af(a) \geq a^*$ is satisfied in the limit. Unfortunately, a simple and general global sufficient condition involving the primitives of the model (a, n, k and function f) is not available. However, numerical calculations suggest that for a wide range of examples with non-decreasing density functions, the local conditions identified in Theorem 1 are sufficient. Our conjecture is that for *all* distributions with non-decreasing density functions this holds true.[10]

[10] It is easy to provide examples with decreasing density function where there is a profitable global deviation and thus a regular equilibrium does not exist. This is not very surprising, since when f is decreasing the virtual utilities may be non-monotone, in which case losing visits from higher types may be less costly for a seller than losing visits from types close to a.