Reflecting inequality of claims in gains and losses

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Abstract

We consider the problem of dividing a resource among a group of agents who have conflicting claims on it. A typical situation is when a firm goes bankrupt and its liquidation value has to be divided among its creditors. We follow the axiomatic approach. Our main axioms are inequality preservation in gains and inequality preservation in losses (Hougaard and Thorlund-Petersen, 2001; Hougaard and Østerdal, 2005). We characterize the family of rules that are continuous, order preserving in gains and losses, and inequality preserving in gains and losses: (i) for two agents, these axioms hardly restrict rules (in particular, all central rules considered in the literature satisfy them); (ii) for three agents, they are significantly more restrictive but they are satisfied by a family containing a continuum of rules; and (iii) for more than three agents, they single out the proportional rule; this result confirms the central role of this rule and furthers our understanding of it in claims problems.

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1 Introduction

How should a resource be divided among a group of agents when they have conflicting claims on it? A typical situation is when a firm goes bankrupt and its liquidation value has to be divided among its creditors. We require that each agent receive a non-negative amount that is at most as large as her claim, and that the individual amounts received by the agents add up to the amount to distribute, the endowment.

An alternative problem that fits this mathematical structure is that of collecting an amount of a resource from a group of agents whose individual endowments add up to at least the amount required. Here, an illustration is a government taxing incomes in order to implement a public project.

We use the language that fits our first example, i.e., each agent has a "claim" on an "endowment" and receives an "award." We refer to these problems as claims problems.[1]

A rule is a function that associates with each problem, i.e., each configuration of claims and endowment, an awards vector for it. In our study of rules, we follow the axiomatic approach.[2] Our first and central two axioms pertain to the class of "distributional" properties of rules (this class has been the object of much attention in recent literature, e.g., Hougaard and Thorlund-Petersen, 2001; Hougaard and Østerdal, 2005; Moreno-Ternero and Villar, 2006a,b; Ju and Moreno-Ternero, 2006, 2008; Bosmans and Lauwers, 2007; and Thomson, 2007).

Our first requirement is that inequalities in claims be reflected in the awards. We adopt the Lorenz criterion, which is commonly used to evaluate income distributions, as the indicator of inequality. Fix the endowment and suppose that two claims vectors, whose coordinates add up to the same amount, are related by Lorenz domination. We require that the awards vectors associated to these claims vectors be also related by Lorenz domination (of course, in the same direction). This axiom, "inequality preservation in gains," introduced by Hougaard and Thorlund-Petersen (2001), is studied by Hougaard and Østerdal (2005). For taxation, the restatement of this axiom is also clear. It requires that if one pre-tax income vector is more equally distributed than another, then the tax vector associated with the more equally distributed income vector be also more equally distributed.

Analogously to inequality preservation in gains, our second requirement is that inequality in claims be reflected in losses. An agent's "loss" at an awards vector is the amount by which her award falls short of her claim. Again, fix the endowment and suppose that two claims vectors, whose coordinates add up to the same amount, are related by Lorenz domination. We require that the losses vectors associated to these claims vectors be also related by Lorenz domination. This axiom, "inequality preservation in losses," also intro-

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[1] Alternative names in the literature are "bankruptcy problems," "taxation problems," and "rationing problems."

[2] O'Neill (1982) pioneered the application of the axiomatic approach to claims problems – see Moulin (2002) and Thomson (2003, 2008) for surveys.

duced by Hougaard and Thorlund-Petersen (2001), is studied by Hougaard and Østerdal (2005).

We study the implications of these two axioms in the presence of three basic additional axioms. First is "order preservation in gains" (Aumann and Maschler, 1985): if agent i's claim is at least as large as agent j's claim, then agent i's award should be at least as large as agent j's award. Second is "order preservation in losses" (Aumann and Maschler, 1985): if agent i's claim is at least as large as agent j's claim, then agent i's loss should be at least as large as agent j's loss. Finally, is continuity: small changes in the data of the problem should not have a large effect on the resulting awards vector. We refer to the first four axioms, i.e., inequality preservation in gains and losses and order preservation in gains and loses, as the "core axioms."

We begin, in Section 4, by providing key structural properties of rules satisfying the core axioms. Then, we characterize the family of rules satisfying continuity and the core axioms. We divide the presentation of this characterization into three cases: two agents, three agents, and more than three agents.

Section 5 presents the characterization in the two-agent case. Our main result here is that continuity and the core axioms, actually, hardly restrict rules. In particular, all central rules considered in the literature, i.e., the constrained equal awards rule, the constrained equal losses rule, concede-and-divide, and the proportional rule, are in the family. In fact, virtually all continuous rules are in this family.

Section 6 presents the characterization in the three-agent case. Here, continuity and the core axioms are significantly more restrictive. Among the central rules, only the proportional rule is admissible. Nevertheless, this family still contains a continuum of rules. We show that, in this family, there is a Lorenz maximal rule, i.e., a rule that Lorenz dominates all others. Moreover, it is the unique Lorenz maximal element. Also, in this family, there is a Lorenz minimal rule, i.e., a rule that is Lorenz dominated by all others. Moreover, it is the unique Lorenz minimal element.[3]

Finally, in Section 7, we characterize the family of rules satisfying the core axioms for more than three agents. Now, these axioms –note that here continuity is not imposed– single out the proportional rule.

Our characterizations show how the core axioms become more and more demanding as the number of agents increases. Most interesting is that when the number of agents reaches the threshold of four, only the proportional rule survives.

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[3] Similar results hold for the two-agent case. Within the family we characterize (Section 5), the constrained equal awards rule is maximal in the Lorenz domination ranking; moreover, it is the unique Lorenz maximal element. Also, within this family, the constrained equal losses rule is minimal in the Lorenz domination ranking; moreover, it is the unique Lorenz minimal element. We omit the presentation of these results in the paper. See Moreno-Ternero and Villar (2006a), Bosmans and Lauwers (2007), and Thomson (2007) for results on Lorenz rankings of rules.

2 Related literature

Our main purpose is to characterize the rules satisfying continuity and the core axioms. This question is first asked by Hougaard and Østerdal (2005). They do not consider the two-agent case and assert that for more than two agents, the proportional rule is the only rule satisfying these five axioms. However, a step in their proof contains a subtle error.[4] In fact, their assertion itself is incorrect for the three-agent case. This is an implication of our characterization result for the three-agent case (Section 6). For more than three agents, Hougaard and Østerdal's assertion is indeed true (Section 7). However, our proofs (Sections 4 and 7) show that a considerably more intricate argument is needed to establish it. Moreover, since our result for that case does not involve continuity, a stronger result holds: the core axioms single out the proportional rule. The complications for that case do not come from our dropping continuity. One can easily see that they remain even if we impose this axiom.

In the context of claims problems, a growing literature has investigated the implications of precluding certain forms of coalitional manipulation. The basic axiom is "no advantageous transfer," i.e., no group of agents should receive more in the aggregate by transferring claims among themselves. The result is that, for more than two agents, this axiom singles out the proportional rule (Moulin, 1985; Chun, 1988; Ju et al., 2007; Ju, 2007; and Thomson, 2008).[5]

Our core axioms are closely related to this form of immunity under coalitional manipulation. They imply a weak form of no advantageous transfer. Suppose that the agents are arranged in the order of increasing claims. Imagine that the "k poorest agents" or the "l richest agents" form an association to protect their rights. It is plausible that these coalitions of agents with similar characteristics form – one can expect large claimants, for instance investment banks, to "talk to each other;" one can also expect small claimants to organize so as to maximize their awards; but, it is less probable that a large claimant will enter into an agreement with a particular small claimant. Now, suppose that the members of such a coalition transfer claims among themselves in such a way that the order of their claims remains the same. Our core axioms imply that the coalition cannot benefit from this manipulation, and our proofs (Sections 4 and 7) implicitly reveal that this weak form of immunity under coalitional manipulation, together with continuity and "equal treatment

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[4] Their proof fails in their Step 3. They write "let c''_k be the highest c_k for which there exists a pair (c'_k, c_k) satisfying $e_k(c'_k) = 0$, $e_k(c_k) = 0$, $e_k(\widehat{c}_k) > 0$, for all $c'_k < \widehat{c}_k < c_k$ (p.359 lines 9-10)." However, the variable c''_k in their statement may not exist, and in fact, examples can be constructed in which it does not exist. Thus their proof fails for any number of agents greater than two.

[5] It is referred as "no advantageous reallocation" in Moulin (1985) and Chun (1988), and "reallocation-proofness" in Ju et al. (2007) and Ju (2007). Another closely related axiom is "summation independence" which says that the award to each agent should only depend on her claim, the total claim, and the amount of resource (Moulin, 1988; Ching and Kakkar, 2001; and Thomson, 2008). It is referred as "decentralizability" in Moulin (1988) and "no-arbitrage" in Ching and Kakkar (2001). In fact, summation independence is equivalent to no advantageous transfer in claims problems (Thomson, 2008).

of equals," characterize the proportional rule for more than three agents.[6]

This result is interesting for two reasons. First, the formation of a coalition and the scope of the manipulation are restricted in our formulation of no advantageous transfer; nevertheless, it essentially characterizes the proportional rule. Second, manipulating coalitions emerge "endogenously" from the data of the problem. The case in which the possible manipulating coalitions are provided as additional data to the problem, in the form of connected components on a network, is studied by Ju (2007).

In the public finance literature, Moyes (1989, 1994) and Arnold (1990) analyze the class of functions that "transform" an income distribution into another one, assuming that each agent's final income depends only on her initial income.[7] The difference between each agent's initial and final incomes is that agent's tax (or subsidy). However, unlike for our model, the total tax to collect (or the total subsidy to distribute) is not exogenously given. In that context, Moyes (1989) and Arnold (1990) show that if there are at least two agents, affine transformations are the only transformations that preserve inequality (in the Lorenz sense) of income distributions. Moyes (1989, 1994) further studies the class of functions that preserve inequality of income distributions for a variety of indicators of inequality (see also Ebert, 2004 for related results). In our model, which includes the specification of how much is available (in theirs, this would be how much to be collected or distributed), their assumption that each agent's final income depends only on her initial income corresponds to assuming that the award to each agent depends only on her claim, the total claim, and the amount available. [8] This assumption itself singles out the proportional rule for more than two agents.[9]

The remainder of the paper is organized as follows. Section 3 presents the model. Section 4 presents structural results. Sections 5, 6, and 7 present our characterizations for two agents, three agents, and more than three agents, respectively. Section 8 discusses extensions of our results. The Appendix contains proofs.

3 The model

3.1 Environment

Let $\boldsymbol{N} \equiv \{1, \ldots, n\}$ be a set of agents. A **claims problem for $\boldsymbol{N}$** is a pair (c, E) where $c = (c_i)_{i \in N} \in \mathbb{R}^N_+$ and $E \in \mathbb{R}_+$ are such that $\sum_{i \in N} c_i \geq E$. The list c is interpreted as a vector of **claims** agents hold and the number E as the amount available of an infinitely

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[6] "Equal treatment of equals" says that two agents with equal claims should receive equal awards.

[7] For each income distribution $x \in \mathbb{R}^n_+$, a "transformation function" $T : \mathbb{R}_+ \to \mathbb{R}_+$ gives another income distribution $(T(x_1), T(x_2), \ldots, T(x_n))$. There are no restriction on T such as $\sum_{i \in N} x_i = \sum_{i \in N} T(x_i)$. Thus, for instance, the following form of a transformation function is admissible: for each $r \in \mathbb{R}_+$, $T(r) = a$ for some $a \in \mathbb{R}_+$ (constant transformation).

[8] This is the axiom of summation independence discussed in Footnote 5.

[9] See Moulin (1988), Ching and Kakkar (2001), and Thomson (2008).

divisible resource, the **endowment**; this amount is not sufficient to honor all claims. The **set of claims problems** is denoted $\boldsymbol{\mathcal{C}}$.

Let $(c,E)\in\mathcal{C}$. An **awards vector for (c,E)** is a vector $x\in\mathbb{R}^N_+$, such that $x\leq c$ and $\sum_{i\in N}x_i=E$.[10] A **rule** is a function that associates with each problem an awards vector for it. The generic rule is denoted S.

For each $c\in\mathbb{R}^N_+$, let $\bar{\boldsymbol{c}}\equiv\sum_{i\in N}c_i$, $\mathbf{min}\,\boldsymbol{c}\equiv\min_{i\in N}c_i$, and $\mathbf{max}\,\boldsymbol{c}\equiv\max_{i\in N}c_i$. For each $s\in\mathbb{R}_+$, let $\boldsymbol{\mathcal{C}(s)}\equiv\{(c,E)\in\mathcal{C}:\bar{c}=s\}$.

3.2 Axioms

A rule S is **continuous** if for each convergent sequence of problems $\{(c_k,E_k)\}_{k\in\mathbb{N}}$ the sequence $\{S(c_k,E_k)\}_{k\in\mathbb{N}}$ converges to $S(\lim_{k\to\infty}(c_k,E_k))$.

The following two properties are introduced by Aumann and Maschler (1985). A rule S is **order preserving in gains (o-p-g)**, if for each $(c,E)\in\mathcal{C}$ and each $\{i,j\}\subseteq N$ such that $c_i\leq c_j$, $S_i(c,E)\leq S_j(c,E)$; it is **order preserving in losses (o-p-l)** if for each $(c,E)\in\mathcal{C}$ and each $\{i,j\}\subseteq N$ such that $c_i\leq c_j$, $c_i-S_i(c,E)\leq c_j-S_j(c,E)$. Let us remark that if a rule S is *o-p-g* or *o-p-l*, then it satisfies **equal treatment of equals (e-t-e)**, i.e., for each $(c,E)\in\mathcal{C}$ and each $\{i,j\}\subseteq N$ such that $c_i=c_j$, $S_i(c,E)=S_j(c,E)$.

For each $y\in\mathbb{R}^N_+$ and each $t\in\{1,\ldots,n\}$, let $y_{[t]}$ be the t-order statistic of y, i.e., the vector $(y_{[t]})_{t=1}^n$ is a permutation of y such that $y_{[1]}\leq y_{[2]}\leq\cdots\leq y_{[n]}$. Let $\{y,y'\}\subseteq\mathbb{R}^N_+$ be such that $\sum_{i\in N}y_i=\sum_{i\in N}y'_i$. Then, $\boldsymbol{y}$ **Lorenz dominates** $\boldsymbol{y'}$, denoted $y\succeq_L y'$, if for each $k\in\{1,\ldots,n\}$, $\sum_{t=1}^k y_{[t]}\geq\sum_{t=1}^k y'_{[t]}$.[11]

The following two properties are introduced by Hougaard and Thorlund-Petersen (2001).

First, if a claims vector Lorenz dominates another, then, for each endowment, the awards vector associated with the first claims vector should also Lorenz dominate the awards vector associated with the second claims vector. Formally, a rule S is **inequality preserving in gains (i-p-g)** if for each $s\in\mathbb{R}_+$ and each $\{(c,E),(c',E)\}\subset\mathcal{C}(s)$ such that $c\succeq_L c'$, $S(c,E)\succeq_L S(c',E)$.

Second, if a claims vector Lorenz dominates another, then, for each endowment, the losses vector associated with the first claims vector should also Lorenz dominate the losses vector associated with the second claims vector. Formally, a rule S is **inequality preserving in losses (i-p-l)** if for each $s\in\mathbb{R}_+$ and each $\{(c,E),(c',E)\}\subset\mathcal{C}(s)$ such that $c\succeq_L c'$, $c-S(c,E)\succeq_L c'-S(c',E)$.

We refer to *o-p-g*, *o-p-l*, *i-p-g*, and *i-p-l* as the **core axioms**.

Let S and S' be two rules. Then, $\boldsymbol{S}$ **Lorenz dominates** $\boldsymbol{S'}$, denoted $S\succeq_L S'$, if for each $(c,E)\in\mathcal{C}$, $S(c,E)\succeq_L S'(c,E)$.

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[10] Vector inequalities: $x\geq y$ allows x and y to be equal; $x\gneq y$ does not; $x>y$ means that each coordinate of x is greater than the corresponding coordinate of y.

[11] See Dalton (1920) and Marshall and Olkin (1979) for more information regarding the Lorenz criterion.

3.3 Central rules

We provide examples of rules that will be mentioned in the following sections. They are central in the literature.

Proportional rule, $\boldsymbol{P}$: For each $(c, E) \in \mathcal{C}$, $P(c, E) \equiv \lambda c$, where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i\in N} P_i(c, E) = E$. This rule divides the endowment proportionally to claims.

Constrained equal awards rule, $\boldsymbol{CEA}$: For each $(c, E) \in \mathcal{C}$ and each $i \in N$, $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i\in N} CEA_i(c, E) = E$. This rule divides the endowment equally subject to no agent receiving more than her claim.

Constrained equal losses rule, $\boldsymbol{CEL}$: For each $(c, E) \in \mathcal{C}$ and each $i \in N$, $CEL_i(c, E) \equiv \max\{c_i - \lambda, 0\}$, where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i\in N} CEL_i(c, E) = E$. This rule divides the aggregate loss (the difference $\sum_{i\in N} c_i - E$) equally subject to no agent receiving a negative amount.

Concede-and-divide, $\boldsymbol{CD}$: For $|N| = 2$. For each $(c, E) \in \mathcal{C}$ and each $i \in N$, $CD_i(c, E) \equiv \max\{E - c_j, 0\} + \frac{E - \sum_{k\in N} \max\{E - c_k, 0\}}{2}$. It can be seen as a hybrid of the constrained equal awards rule and the constrained equal losses rule. If the endowment is less than or equal to $\bar{c}/2$, then the constrained equal award rule is applied. If the endowment is greater than $\bar{c}/2$, each agent first receives her half-claim, then the constrained equal losses rule is applied to divide the rest.[12]

4 Structural results

In this section we provide key structural properties of rules satisfying the *core axioms.*

Our first result is that for each of these rules, the award to a minimal claimant is a function of her claim, the aggregate claim, and the endowment. We call this function the "minimal award function" (associated with the rules). A symmetric statement is also true for the award of a maximal claimant, i.e., her award is a function of her claim, the aggregate claim, and the endowment. Let i be a maximal claimant in problem (c, E). Then, i's award in problem (c, E) can be calculated in terms of the minimal award function, by means of a residual award argument as follows: it is the award for this claimant in the problem with aggregate claim $\bar{c}$, endowment E, and in which i has claim c_i and all the other agents are minimal claimants.[13] The following lemma formalizes these results.

Let $\boldsymbol{Y_n} \equiv \{(r, s, E) \in \mathbb{R}^3_+ : r \le s/n \text{ and } s \ge E\}$.

Lemma 1. Assume $n \ge 2$. If a rule S satisfies the *core axioms*, then there is $m : Y_n \to \mathbb{R}_+$ such that for each $(c, E) \in \mathcal{C}$ and each $i \in \arg\min_{k\in N} c_k$,

$$S_i(c, E) = m(\min c, \bar{c}, E).$$

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[12] This rule is introduced by Aumann and Maschler (1985) and its name is suggested by Thomson (2003).

[13] One could take the function that determines the award for a maximal claimant as primitive and identify the minimal one by a symmetric argument.

Moreover, for each $(c, E) \in \mathcal{C}$ and each $i \in \arg\max_{k \in N} c_k$,

$$S_i(c, E) = E - (n-1)\, m\!\left(\frac{\bar{c} - \max c}{n-1}, \bar{c}, E\right).$$

Proof. Let S be a rule satisfying the *core axioms*. We proceed in three steps. The statements and proof of Steps 1 and 2 are similar to arguments in Hougaard and Østerdal (2005).

Step 1: For each $s \in \mathbb{R}_+$, each $\{(c, E), (c', E)\} \subset \mathcal{C}(s)$ such that $\min c = \min c'$, each $i \in \arg\min_{k \in N} c_k$, and each $j \in \arg\min_{k \in N} c'_k$, $S_i(c, E) = S_j(c', E)$.

Let $s \in \mathbb{R}_+$ and $\{(c, E), (c', E)\} \subset \mathcal{C}(s)$ be such that $\min c = \min c'$. Let $i \in \arg\min_{k \in N} c_k$ and $j \in \arg\min_{k \in N} c'_k$. We prove that $S_i(c, E) = S_j(c', E)$. Let $c^* \in \mathcal{C}(s)$ be the vector defined by: $c^*_i \equiv \min c$, and for each $l \in N \setminus \{i\}$, $c^*_l \equiv \frac{s - \min c}{n-1}$. Observe that $\min c = \min c^* = \min c'$, $c \preceq_L c^*$, and $c^* \succeq_L c'$. Then by *i-p-g*, $S_{[1]}(c, E) \leq S_{[1]}(c^*, E)$ and $S_{[1]}(c^*, E) \geq S_{[1]}(c', E)$. By *o-p-g*, $S_{[1]}(c, E) = S_i(c, E)$, $S_{[1]}(c^*, E) = S_i(c^*, E)$, and $S_{[1]}(c', E) = S_j(c', E)$. Thus, $S_i(c, E) \leq S_i(c^*, E)$ and $S_i(c^*, E) \geq S_j(c', E)$. By *i-p-l*, $[c - S(c, E)]_{[1]} \leq [c^* - S(c^*, E)]_{[1]}$ and $[c^* - S(c^*, E)]_{[1]} \geq [c' - S(c', E)]_{[1]}$. By *o-p-l*, $[c - S(c, E)]_{[1]} = c_i - S_i(c, E)$, $[c^* - S(c^*, E)]_{[1]} = c^*_i - S_i(c^*, E)$, and $[c' - S(c', E)]_{[1]} = c'_j - S_j(c', E)$. Thus, $\min c - S_i(c, E) \leq \min c^* - S_i(c^*, E)$ and $\min c^* - S_i(c^*, E) \geq \min c' - S_j(c', E)$. Altogether, $S_i(c, E) = S_i(c^*, E) = S_j(c', E)$.

Step 2: For each $s \in \mathbb{R}_+$, each $\{(c, E), (c', E)\} \subset \mathcal{C}(s)$ such that $\max c = \max c'$, each $i \in \arg\max_{k \in N} c_k$, and each $j \in \arg\max_{k \in N} c'_k$, $S_i(c, E) = S_j(c', E)$. We omit the proof since it is parallel to that of Step 1.

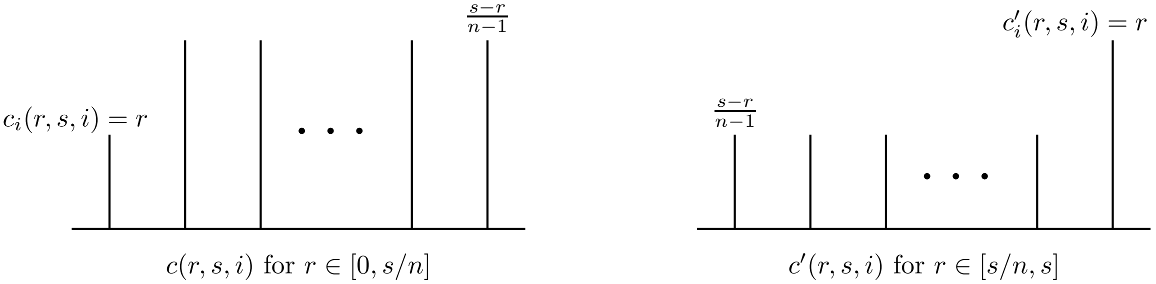


Figure 1: Illustrating Step 3 (Lemma 1).

Step 3: Determining the awards to minimal and maximal claimants.

Figure 1 illustrates this step.

Minimal claimants. Let $s \in \mathbb{R}_+$, $r \in [0, s/n]$, and $i \in N$. Let $c(r, s, i) \in \mathcal{C}(s)$ be the vector defined by: $c_i(r, s, i) \equiv r$ and for each $j \in N \setminus \{i\}$, $c_j(r, s, i) \equiv \frac{s-r}{n-1}$. Let $m : Y_n \to \mathbb{R}_+$ be the function defined as follows: for each $(r, s, E) \in Y_n$, $m(r, s, E) \equiv S_i(c(r, s, i), E)$ for some $i \in N$. By Step 1 and since $\min c(r,s,i) = r$, then m is well-defined, and for each $(c,E) \in \mathcal{C}$ and each $i \in \arg\min_{k\in N} c_k$, $S_i(c,E) = m(\min c, \bar{c}, E)$.

Maximal claimants. Let $s \in \mathbb{R}_+$, $r \in [s/n, s]$, and $i \in N$. Let $c'(r,s,i) \in \mathcal{C}(s)$ be the vector defined by: $c'_i(r,s,i) \equiv r$ and for each $j \in N \setminus \{i\}$, $c_j(r,s,i) \equiv \frac{s-r}{n-1}$. Since $\min c'(r,s,i) = \frac{s-r}{n-1}$, then for each $E \in \mathbb{R}_+$ such that $E \leq s$, and each $j \in N \setminus \{i\}$, $S_j(c'(r,s,i),E) = m\left(\frac{s-r}{n-1}, s, E\right)$. Thus, $S_i(c'(r,s,i),E) = E - (n-1)m\left(\frac{s-r}{n-1}, s, E\right)$. By Step 2 and since $\max c'(r,s,i) = r$, then for each $(c,E) \in \mathcal{C}$ and each $i \in \arg\max_{k\in N} c_k$,

$$S_i(c,E) = E - (n-1)\, m\!\left(\frac{\bar{c} - \max c}{n-1}, \bar{c}, E\right).$$

□

Our second result, Lemma 2, is a partial characterization of minimal award functions. It states four properties satisfied by each minimal award function associated to a rule that satisfies the core axioms. The third property plays an important role in our characterizations when there are three or more agents (Sections 6 and 7).

Lemma 2. Let S be a rule satisfying the *core axioms* and $m : Y_n \to \mathbb{R}_+$ its associated minimal award function. Then, m satisfies the following properties:

(P1) *For each $(s,E) \in \mathbb{R}^2_+$ such that $s \geq E$, $m(\cdot, s, E) : [0, s/n] \to \mathbb{R}_+$ is a non-decreasing function such that $m(0,s,E) = 0$ and $m(s/n, s, E) = E/n$.*

(P2) *For each $(s,E) \in \mathbb{R}^2_+$ such that $s \geq E$, and each $\{r, r'\} \subset [0, s/n]$ such that $r' \geq r$,*

$$m(r', s, E) - m(r, s, E) \leq r' - r.$$

(P3) *For each $(r,s,E) \in Y_n$,*

$$m\!\left(\frac{s + (n-2)r}{2(n-1)}, s, E\right) = \frac{E}{2(n-1)} + \frac{n-2}{2(n-1)} m(r,s,E). \tag{1}$$

(P4) *m is continuous in its first argument; moreover, it is continuous whenever S is.*

Proof. • **P1.** Let $(s,E) \in \mathbb{R}^2_+$ be such that $s \geq E$. Let $\{r, r'\} \subset [0, s/n]$ be such that $r' \geq r$. Let $i \in N$. Let $c(r,s,i) \in \mathcal{C}(s)$ be the vector defined by: $c_i(r,s,i) \equiv r$ and for each $j \in N \setminus \{i\}$, $c_j(r,s,i) \equiv \frac{s-r}{n-1}$. Since $c(r',s,i) \succeq_L c(r,s,i)$ and S satisfies *i-p-g*, then $S(c(r',s,i),E) \succeq_L S(c(r,s,i),E)$. Thus, $m(r',s,E) = S_i(c(r',s,i),E) \geq S_i(c(r,s,i),E) = m(r,s,E)$. Therefore, the function $m(\cdot, s, E)$ is non-decreasing. Since $S_i(c(0,s,i),E) = 0$, then $m(0,s,E) = 0$. Note that $c(s/n, s, i) = (s/n, \cdots, s/n)$. Since S satisfies *e-t-e*, then $S_i(c(s/n,s,i), E) = E/n$. Thus, $m(s/n, s, E) = E/n$.

- **P2.** Let $(s,E) \in \mathbb{R}^2_+$ be such that $s \geq E$. Let $\{r, r'\} \subset [0, s/n]$ be such that $r' \geq r$. Let $i \in N$. Since $c(r', s, i) \succeq_L c(r, s, i)$ and S satisfies *i-p-l*, then

$$c(r', s, i) - S(c(r', s, i), E) \succeq_L c(r, s, i) - S(c(r, s, i), E).$$

Thus, $r' - m(r', s, E) = r' - S_i(c(r', s, i), E) \geq r - S_i(c(r, s, i), E) = r - m(r, s, E)$. Therefore, $m(r', s, E) - m(r, s, E) \leq r' - r$.

- **P3.** Clearly, P1 implies P3 when $n = 2$. Assume now that $n \geq 3$. Let $(r, s, E) \in Y_n$. We prove (1). Note that $r \in [0, s/n]$. Let $c^* \in \mathcal{C}(s)$ be the vector defined by: for each $i \leq n-2$, $c^*_i \equiv r$, and for each $i \geq n-1$, $c^*_i \equiv \frac{s-(n-2)r}{2}$. Figure 2 illustrates this construction. Since $\min c^* = r$, then by Lemma 1, $S_{n-2}(c^*, E) = m(r, s, E)$. Now, since $\max c^* = \frac{s-(n-2)r}{2}$, then by Lemma 1,

$$S_{n-2}(c^*, E) = E - (n-3)m(r, s, E) - 2\left[E - (n-1)\, m\!\left(\frac{s - \frac{s-(n-2)r}{2}}{n-1}, s, E\right)\right].$$

We obtain (1) by equating $S_{n-2}(c^*, E)$ in the two expressions above.

- **P4.** Clearly, if S is continuous, so is m. One easily sees that (P1) and (P2) imply $m(\cdot, s, E)$ is a contraction and thus continuous. □

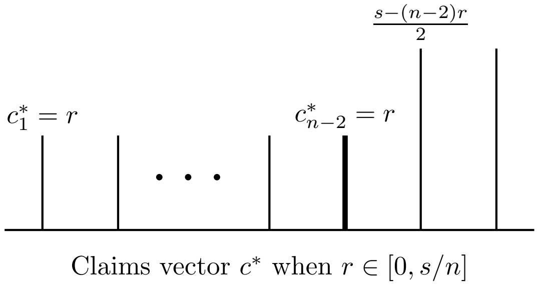


Figure 2: Illustrating proof of P3 in Lemma 2. The award to agent $n-2$ can be derived in two ways. One is to consider her as a minimal claimant. The other is to give her the endowment minus what the others (the minimal claimants and the maximal claimants) receive.

5 Proportional and non-proportional inequality preserving rules for the two-agent case

We assume throughout this section that $n = 2$. The following theorem characterizes the family of rules satisfying *continuity* and the *core axioms*. In particular, it implies that the converse to Lemma 2 also holds. Let us remark that since $n = 2$, P1 implies P3 and thus, this later property does not appear in the statement of the theorem.

Theorem 1. Assume $n = 2$. A rule S satisfies *continuity* and the *core axioms* if, and only if, there is a continuous $m : Y_2 \longrightarrow \mathbb{R}_+$ satisfying P1 and P2 such that for each $(c, E) \in \mathcal{C}$ and each $i \in N$,

$$S_i(c,E) = \begin{cases} m(c_i, \overline{c}, E) & \text{if } c_i = \min c < \max c, \\ E - m(\overline{c} - c_i, \overline{c}, E) & \text{if } c_i = \max c. \end{cases} \tag{2}$$

The proof is in Appendix A.1. Lemmas 1 and 2 imply that for each *continuous* rule that satisfies the *core axioms*, its associated minimal award function is continuous and satisfies P1 and P2. Since this function determines the award of a minimal claimant, it is obvious that the rule has to coincide with the corresponding expression (2). The non-trivial part of the theorem is to establish that given a continuous function $m : Y_2 \longrightarrow \mathbb{R}_+$ satisfying P1 and P2, expression (2) defines a *continuous* rule that satisfies the *core axioms*.

A consequence of Theorem 1 is that for the two-agent case, inequality preservation is easily attained. For each continuous function $m : Y_2 \longrightarrow \mathbb{R}_+$ satisfying P1 and P2, let $\boldsymbol{S^m}$ **be the rule associated with** $\boldsymbol{m}$, i.e., the rule defined by expression (2) (Theorem 1 implies such a rule is well-defined). Since continuity, P1, and P2 are barely restrictive, this class of rules is large.

The following are two examples of continuous minimal award functions satisfying P1 and P2.

Example 1. Proportional minimal award function (two-agent case). Let $m_P : Y_2 \longrightarrow \mathbb{R}_+$ be the function defined by: $m_P(0) \equiv 0$ and for each $(r, s, E) \in Y_2 \setminus \{0\}$, $m_P(r, s, E) \equiv \frac{r}{s}E$. One can easily verify that m_P is continuous and satisfies P1 and P2. $\square$

Example 2. Constrained equal awards minimal award function (two-agent case). Let $m_{CEA} : Y_2 \longrightarrow \mathbb{R}_+$ be the function defined by: for each $(r, s, E) \in Y_2$,

$$m_{CEA}(r,s,E) \equiv \begin{cases} r & \text{if } r \in [0, E/2] \\ E/2 & \text{if } r \in]E/2, s/2]. \end{cases}$$

One can easily verify that m_{CEA} is continuous and satisfies P1 and P2. Moreover, for each $(c, E) \in \mathcal{C}$, $m_{CEA}(\min c, \bar{c}, E)$ is the award to the minimal claimant recommended by the constrained equal awards rule in (c, E). $\square$

It is straightforward to show that $S^{m_P} = P$ and $S^{m_{CEA}} = CEA$. Thus, P and CEA satisfy *continuity* and the *core axioms*. Additionally, so do CEL and CD. Moreover, any convex combination of rules in the family and, in fact, virtually all continuous rules considered so far in the literature do.

6 Proportional and non-proportional inequality preserving rules for the three-agent case

We assume throughout this section that $n = 3$. The following theorem characterizes the family of rules satisfying *continuity* and the *core axioms*. In particular, it implies that in the three-agent case a converse to Lemma 2 also holds.

Theorem 2. Assume $n = 3$. A rule S is *continuous* and satisfies the *core axioms* if, and only if, there is a continuous $m : Y_3 \to \mathbb{R}_+$ satisfying P1, P2, and P3 such that for each $(c, E) \in \mathcal{C}$ and each $i \in N$,

$$S_i(c,E) = \begin{cases} m(c_i, \overline{c}, E) & \text{if } c_i = \min c < \max c, \\ E - m(\min c, \overline{c}, E) - [E - 2m(\frac{\overline{c} - \max c}{2}, \overline{c}, E)] & \text{if } \min c < c_i < \max c, \\ E - 2m(\frac{\overline{c} - c_i}{2}, \overline{c}, E) & \text{if } c_i = \max c. \end{cases} \tag{3}$$

The proof is in Appendix A.2. As for the two-agent case, in the three-agent case Lemmas 1 and 2 imply that for each *continuous* rule that satisfies the *core axioms*, the associated minimal award function is continuous and satisfies P1, P2, and P3. Moreover, since the minimal award function determines the award to a minimal claimant, it is obvious that awards to minimal and maximal claimants are given by expression (3). Then, by the feasibility constraint, the award to an intermediate claimant is also given by this expression. The surprising and non-trivial part of this theorem is the converse. Given a continuous function $m : Y_3 \to \mathbb{R}_+$ that satisfies P1, P2, and P3, it is not even clear that expression (3) defines a rule. But, in fact, it not only does, but it also defines a *continuous* rule that satisfies the *core axioms*.

Our next task is to understand further the class of *continuous* rules that satisfy the *core axioms*, e.g., its further structure and richness. We do so, in the following subsection, by providing a mathematical characterization of all continuous functions $m : Y_3 \to \mathbb{R}_+$ satisfying P1, P2, and P3. We show that this class contains a continuum of functions and a simple geometric construction allows one to span the whole set using as parameter the "initial portion" of them.

6.1 Characterization of minimal award functions in three-agent case

In this subsection we characterize the family of continuous functions $m : Y_3 \to \mathbb{R}_+$ that satisfy P1, P2, and P3.

Properties P1 and P2 are self explanatory. By contrast, it is not clear which functions also satisfy P3. Let $m : Y_3 \to \mathbb{R}_+$ be such a function. We show that P3 imposes only one restriction on m on the "initial portion" of its domain, i.e., the set $\boldsymbol{X} \equiv \{(r, s, E) \in \mathbb{R}^3_+ : r \leq s/4 \text{ and } s \geq E\}$. The requirement is that for each $(s, E) \in \mathbb{R}^2_+$ such that $s \geq E$, the function attain the proportional value at $(s/4, s, E)$, i.e., $m(s/4, s, E) = E/4$. In fact, given any function f defined on X that satisfies P1 and P2 restricted to X and such that for each $(s, E) \in \mathbb{R}^2_+$ such that $s \geq E$, $f(s/4, s, E) = E/4$, there is a unique continuous function $m : Y_3 \to \mathbb{R}_+$ that satisfies P1, P2, and P3 and coincides with f on X. Our main result in this subsection provides an explicit formula, which has a simple geometrical interpretation, that gives for each of these f, its unique extension.

Let us first formally define the family of continuous functions on X that parameterize our construction. The set of **initial minimal award functions**, denoted $\boldsymbol{\mathcal{F}}$, is the set of continuous functions $f : X \to \mathbb{R}_+$ that satisfies the following properties: for each $(s, E) \in \mathbb{R}^2_+$ such that $s \geq E$,

(p1) $f(\cdot, s, E) : [0, s/4] \to \mathbb{R}_+$ *is non-decreasing and* $f(0, s, E) = 0$.

(p2) *for each* $\{r, r'\} \subset [0, s/4]$ *such that* $r' \geq r$, $f(r', s, E) - f(r, s, E) \leq r' - r$.

(a1) $f(s/4, s, E) = E/4$.

Let us remark that in the definition above, properties p1 and p2 are the restrictions of P1 and P2 to X, respectively. Let us also remark that properties p1, p2, and a1 are satisfied by a continuum of functions. Moreover, the class $\mathcal{F}$ is closed under convex combinations.

The following are two examples of functions in $\mathcal{F}$.

Example 3. Proportional initial minimal award function. Let $f_P : X \to \mathbb{R}_+$ be the function defined by: $f_P(0) \equiv 0$ and for each $(r, s, E) \in X \setminus \{0\}$, $f_P(r, s, E) \equiv \frac{r}{s}E$. One can easily verify that $f_P \in \mathcal{F}$. □

Example 4. Ray-generated initial minimal award function. Let $\rho \in]0, 1[$. The ray-generated initial minimal award function associated with ρ is the function, $f_\rho : X \to \mathbb{R}_+$, defined by: for each $(r, s, E) \in X$,

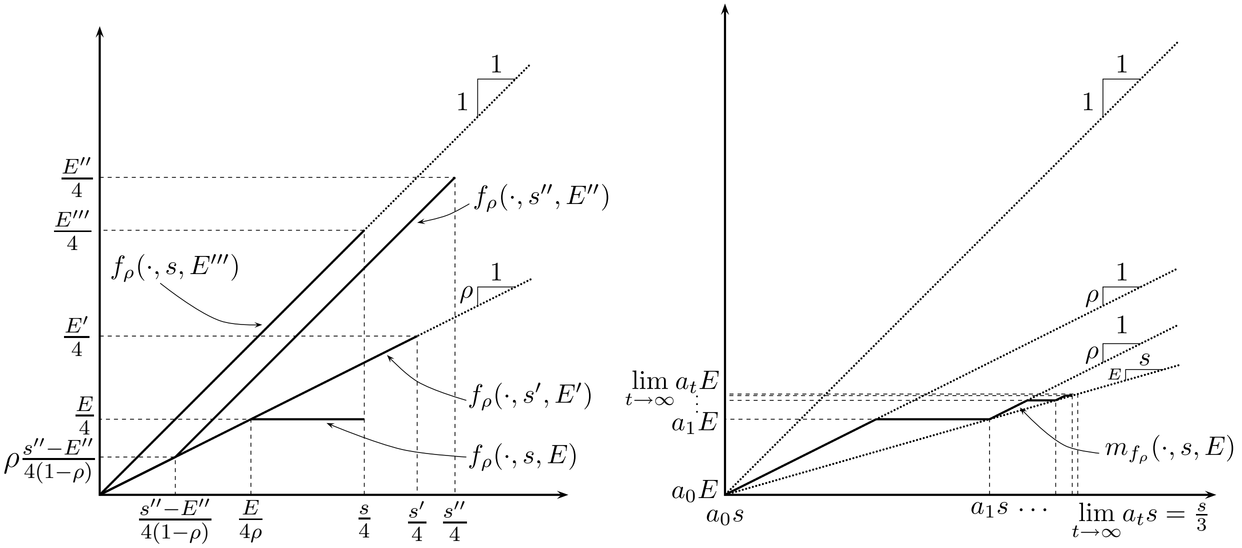
$$f_\rho(r, s, E) \equiv \begin{cases} \rho r & \text{if } r \in [0, \frac{s-E}{4(1-\rho)}] \text{ and } E > \rho s, \\ r - \frac{s-E}{4} & \text{if } r \in]\frac{s-E}{4(1-\rho)}, \frac{s}{4}] \text{ and } E > \rho s, \\ \rho r & \text{if } r \in [0, \frac{E}{4\rho}] \text{ and } E \leq \rho s, \\ \frac{E}{4} & \text{if } r \in]\frac{E}{4\rho}, \frac{s}{4}] \text{ and } E \leq \rho s. \end{cases}$$

Figure 3(a) displays the graph of $f_\rho(\cdot, s, E)$ for different values of (s, E). One can easily verify that $f_\rho \in \mathcal{F}$. □

We now describe how to construct the unique continuous extension of each $f \in \mathcal{F}$ satisfying P1, P2, and P3. We denote it $\boldsymbol{m_f}$. Let $(s, E) \in \mathbb{R}^2_+$ be such that $s \geq E$. Obviously, $m_f(\cdot, s, E)$ and $f(\cdot, s, E)$ coincide on $[0, s/4]$. Now, for $]s/4, s/4 + s/4^2]$, $m_f(\cdot, s, E)$ coincides with the function obtained by homothetic contraction of $f(\cdot, s, E)$ to the interval $[0, s/4^2]$ and translation of the origin to $(s/4, E/4)$, i.e., for each $r \in]s/4, s/4 + s/4^2]$,

$$m_f(r, s, E) = \frac{E}{4} + \frac{1}{4} f\left(4\left(r - \frac{s}{4}\right), s, E\right).$$

That is, $m_f(\cdot, s, E)$ is obtained by concatenating $f(\cdot, s, E)$ and its homothetic contraction to $[0, s/4^2]$. Figure 3 (b) illustrates this construction for f_ρ in Example 4. For $]s/4+s/4^2, s/4+s/4^2+s/4^3]$, $m_f(\cdot, s, E)$ coincides with the function obtained by homotheticly contracting $f(\cdot, s, E)$ to the interval $[0, s/4^3]$ and translating the origin to $(s/4 + s/4^2, E/4 + E/4^2)$. The construction is recursively completed by concatenating the subsequent homothetic contractions of $f(\cdot, s, E)$ in this way. Since $\lim_{t\to\infty} \sum_{j=1}^{t} s/4^j = s/3$, this process completes the definition of $m_f(\cdot, s, E)$ for $[0, s/3[$. Finally, we set $m_f(s/3, s, E)$ to be $E/3$.



(a) (b)

Figure 3: Initial minimal award and minimal award functions; (a) Example 4, f_ρ; (b) m_{f_ρ}.

We now present our theorem formally. Let $\{a_t\}_{t\in\{0\}\cup\mathbb{N}}$ be the sequence defined as follows: for each $t \in \{0\} \cup \mathbb{N}$, $a_t \equiv (1 - 1/4^t)/3$. Let us remark that for each $t \in \mathbb{N}$, $a_t \equiv \sum_{j=1}^{t} 1/4^j$, i.e., this is the sequence of endpoints of intervals obtained by adding subsequent intervals of lengths equal to the powers of 1/4.

Proposition 1. Assume $n = 3$. A continuous function $m : Y_3 \to \mathbb{R}_+$ satisfies P1, P2, and P3 if, and only if, there is $f \in \mathcal{F}$ such that m coincide with the function defined by: for each $(r, s, E) \in Y_3$,

$$m_f(r, s, E) \equiv \begin{cases} 0 & \text{if } r = 0, \\ a_t E + \frac{1}{4^t} f\left(4^t(r - a_t s), s, E\right) & \text{if there is } t \in \{0\} \cup \mathbb{N} \text{ s.t. } r \in]a_t s, a_{t+1} s], \\ \frac{E}{3} & \text{if } r = \frac{s}{3}. \end{cases}$$

Moreover, if a continuous function $m' : Y_3 \to \mathbb{R}_+$ satisfies P1, P2, P3, and coincides with m on X, then $m = m'$.

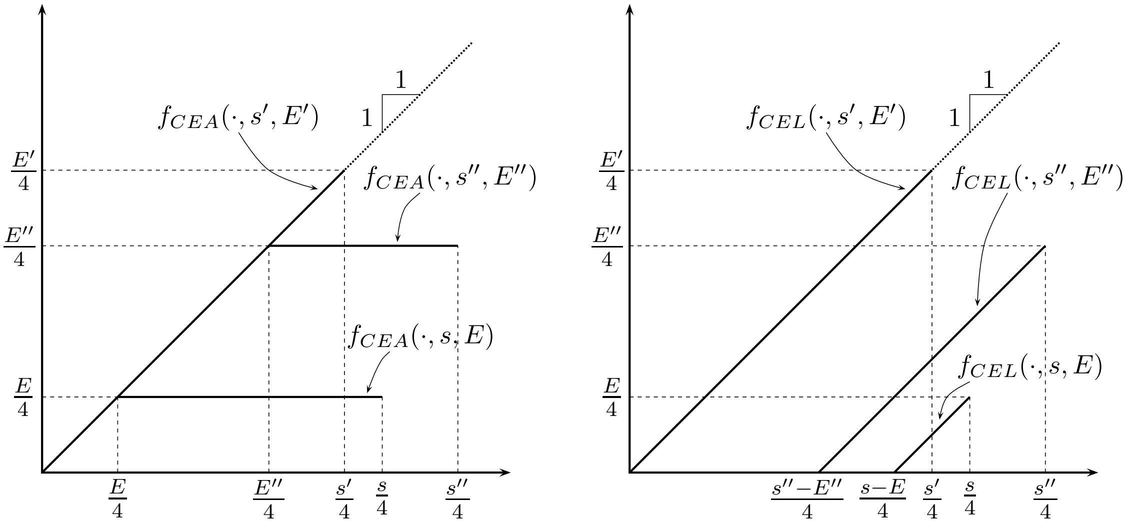
The proof is in Appendix A.2.

We now can provide an alternative characterization of the class of *continuous* rules that satisfy the *core axioms* in the three-agent case. For each $f \in \mathcal{F}$, let $\boldsymbol{S^f}$ be the rule defined by expression (3) for the minimal award function m_f. The following proposition is a direct consequence of Theorem 2 and Proposition 1.

Proposition 2. Assume $n = 3$. A rule S is *continuous* and satisfies the *core axioms* if, and only if, there is $f \in \mathcal{F}$ such that $S = S^f$.

6.2 Lorenz extreme rules

In this subsection we introduce two additional *continuous* rules that satisfy the *core axioms*. These rules "follow" CEA and CEL respectively, as closely as possible, given the restrictions implied by these axioms. We show that they are Lorenz extremes in this class.



(a) (b)

Figure 4: Initial minimal award functions: (a) Example 5; (b) Example 6.

Example 5. Constrained equal awards initial minimal award function. Let $f_{CEA} : X \to \mathbb{R}_+$ be the function defined as follows: for each $(r, s, E) \in X$,

$$f_{CEA}(r, s, E) \equiv \begin{cases} r & \text{if } r \in [0, E/4] \\ E/4 & \text{if } r \in]E/4, s/4]. \end{cases}$$

One easily sees that $f_{CEA} \in \mathcal{F}$. Let $(c, E) \in \mathcal{C}$ be such that $\min_{i \in N} c_i \leq E/4$. A special feature of f_{CEA} is that $f_{CEA}(\min c, \bar{c}, E)$ is the award to the minimal claimant recommended by CEA in (c, E). Figure 3(a) displays the graph of $f_{CEA}(\cdot, s, E)$ for different values of (s, E). $\square$

Example 6. Constrained equal losses initial minimal award function. Let $f_{CEL} : X \rightarrow \mathbb{R}_+$ be the function defined as follows: for each $(r, s, E) \in X$,

$$f_{CEL}(r, s, E) \equiv \begin{cases} 0 & \text{if } r \in [0, \frac{s-E}{4}] \\ r - \frac{s-E}{4} & \text{if } r \in]\frac{s-E}{4}, \frac{s}{4}]. \end{cases}$$

One easily sees that $f_{CEL} \in \mathcal{F}$. Let $(c, E) \in \mathcal{C}$ be such that $\min_{i \in N} c_i \leq \frac{s-E}{4}$. A special feature of f_{CEL} is that $f_{CEL}(\min c, \bar{c}, E)$ is the award to the minimal claimant recommended by CEL in (c, E). Figure 3(b) displays the graph of $f_{CEL}(\cdot, s, E)$ for different values of (s, E). □

A recent literature has investigated the existence, and provided characterization, of Lorenz maximal and Lorenz minimal rules among different classes of rules (Moreno-Ternero and Villar, 2006a; Bosmans and Lauwers, 2007; and Thomson, 2007). Let $CEA^* \equiv S^{f_{CEA}}$. The next proposition states that CEA^* is, in fact, Lorenz maximal in the class of rules we study.

Proposition 3. Assume $n = 3$. Then, CEA^* is the unique Lorenz maximal rule in the class of *continuous* rules that satisfies the *core axioms.*

The proof is in Appendix A.2.

Now, let $CEL^* \equiv S^{f_{CEL}}$. The next proposition is the counterpart of Proposition 3. It states that CEL^* is, in fact, Lorenz minimal in the class of rules we study.

Proposition 4. Assume $n = 3$. Then, CEL^* is the unique Lorenz minimal rule in the class of *continuous* rules that satisfies the *core axioms.*

We omit the proof since it is parallel to that of Proposition 3.

7 Inequality preserving rules for more than three agents

We assume throughout this section that $n > 3$. We characterize the family of rules that are *continuous* and satisfy the *core axioms.* Unlike for the two-agent and three-agent cases, these axioms single out the proportional rule. In fact, the following theorem states that even without requiring *continuity*, the *core axioms* single out this rule.

Theorem 3. Assume $n > 3$. A rule satisfies the *core axioms* if, and only if, it is the proportional rule.

First, we sketch the proof. Consider a rule satisfying the *core axioms.* From Lemma 1 we know that there is a minimal award function associated with the rule, say m, that satisfies P1 to P4. Our proof consists on showing that m coincides with the minimal award function associated to the proportional rule. Let $c_1 \leq \cdots \leq c_n$ and $c'_1 \leq \cdots \leq c'_n$ be two claims vectors with the same aggregate claim, $\bar{c} \geq E$. Let us omit the second and third arguments of m, with the understanding that they are the fixed values $\bar{c}$ and E. A crucial step of our proof is showing that if c_1 and c_1' are "small and close to each other," then there is $\alpha \in (0,1)$ such that $m(\alpha c_1 + (1-\alpha)c_1') = \alpha m(c_1) + (1-\alpha)m(c_1')$. Since m is continuous in its first argument, this turns out to be enough to guarantee that it is linear in its first argument on $[0, \frac{2\bar{c}}{3(n-1)}]$. Using P3, we extend this result to $[0, \frac{\bar{c}}{n}]$. Since m satisfies P1, then it interpolates the proportional value when $c_1 = \frac{\bar{c}}{n}$. Thus, it coincides with the proportional minimal award function.

Let us remark that our proof relates only problems with equal aggregate claims and equal endowments. Since the core axioms imply continuity in the first argument of m, then no additional notion of continuity is necessary to imply proportionality.

Proof. It is easy to see that the proportional rule satisfies the properties listed in the theorem. Conversely, let S be a rule satisfying these properties. We prove that $S = P$. Let $m : Y_n \to \mathbb{R}_+$ be the function whose existence is stated in Lemma 1. Recall that for each $(c, E) \in \mathcal{C}$, each $i \in \arg\min_{k\in N} c_k$, and each $j \in \arg\max_{k\in N} c_k$,

$$S_i(c,E) = m(\min c, \bar{c}, E) \text{ and } S_j(c,E) = E - (n-1)\, m\!\left(\frac{\bar{c} - \max c}{n-1}, \bar{c}, E\right). \tag{4}$$

By Lemma 2, m satisfies property P3, i.e, for each $(r, s, E) \in Y_n$,

$$m\!\left(\frac{s + (n-2)r}{2(n-1)}, s, E\right) = \frac{E}{2(n-1)} + \frac{n-2}{2(n-1)} m\left(r, s, E\right). \tag{5}$$

We complete the proof in six steps. Step 1 is similar to an argument in Hougaard and Østerdal (2005).

Step 1: Let $(s, E) \in \mathbb{R}^2_+$ be such that $s \geq E$, $\{(c, E), (c', E)\} \subset \mathcal{C}(s)$ such that $c_1 \leq \cdots \leq c_n$ and $c_1' \leq \cdots \leq c_n'$, and $1 \leq t \leq n$.

(a) If for each $i \leq t$, $c_i = c_i'$, then for each $i \leq t$, $S_i(c, E) = S_i(c', E)$.

(b) If for each $i \geq t$, $c_i = c_i'$, then for each $i \geq t$, $S_i(c, E) = S_i(c', E)$.

First, we prove (a). Let s, E, $\{(c, E), (c', E)\}$, and t be as in the statement above. We prove the assertion by induction on t. If $t = 1$, then by (4), if $c_1 = c_1'$, then $S_1(c, E) = S_1(c', E)$. Let $t \in \{2, \ldots, n\}$. Suppose that if for each $i \leq t-1$, $c_i = c_i'$, then for each $i \leq t-1$, $S_i(c, E) = S_i(c', E)$. Assume that for each $i \leq t$, $c_i = c_i'$. By the induction hypothesis, we already know that for each $i \leq t-1$, $S_i(c, E) = S_i(c', E)$. Thus, it is enough to show that $S_t(c, E) = S_t(c', E)$. Let $c^* \in \mathcal{C}(s)$ be the vector defined as follows: for each $i \leq t$, $c_i^* \equiv c_i$, and for each $j > t$, $c_j^* \equiv \frac{s - \sum_{l \leq t} c_l}{n-t}$. Observe that $c^* \succeq_L c$. Then by *o-p-g* and *i-p-g*, $S_1(c, E) + \cdots + S_t(c, E) \leq S_1(c^*, E) + \cdots + S_t(c^*, E)$. Now, by *o-p-l* and *i-p-l*,

$$[c_1 - S_1(c, E)] + \cdots + [c_t - S_t(c, E)] \leq [c_1^* - S_1(c^*, E)] + \cdots + [c_t^* - S_t(c^*, E)].$$

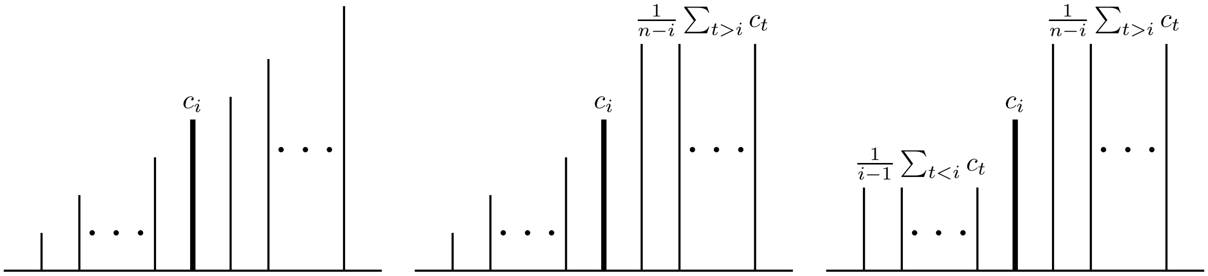
Altogether, $S_1(c, E) + \cdots + S_T(c, E) = S_1(c^*, E) + \cdots + S_t(c^*, E)$. A symmetric argument shows that $S_1(c', E) + \cdots + S_t(c', E) = S_1(c^*, E) + \cdots + S_t(c^*, E)$. Since for each $i \leq t-1$, $S_i(c, E) = S_i(c', E)$, then $S_t(c, E) = S_t(c', E)$.

We omit the proof of (b) since it is parallel to that of (a).

Step 2: Determining the award to an intermediate claimant (Figure 5). Let $c \in \mathbb{R}^N_+$ and suppose that $c_1 \leq \cdots \leq c_n$. Let $i \in N$ be such that $1 < i < n$. Let $c^*(c, i) \in \mathcal{C}(\bar{c})$ be the vector defined as follows: for each $j < i$, $c^*_j(c, i) \equiv \frac{1}{i-1} \sum_{t<i} c_t$, $c^*_i(c, i) \equiv c_i$, and for each $j > i$, $c^*_j(c, i) \equiv \frac{1}{n-i} \sum_{t>i} c_t$. By Step 1, $S_i(c, E) = S_i(c^*(c, i), E)$. By (4),

$$S_i(c,E) = E-(i-1)m\left(\frac{1}{i-1}\sum_{t<i} c_t, \bar{c}, E\right)-(n-i)\left[E-(n-1)m\left(\frac{\bar{c}-\frac{1}{n-i}\sum_{t>i} c_t}{n-1}, \bar{c}, E\right)\right].$$

Let $(s, E) \in \mathbb{R}^2_+$ be such that $s > 0$ and $s \geq E$. Since s and E are fixed throughout Steps 3 to 6, we omit the second and third arguments of m.



(a) Initial claims vector c (b) Intermediate claims vector (c) Claims vector $c^*(c, i)$

Figure 5: Illustrating Step 2 (Theorem 3). Agent i's award is the same for claims vectors in (a), (b), and (c).

Step 3: For each $r \in [0, \frac{s}{n}[$, and each $y \in [r, \frac{2s+(n-3)r}{3(n-1)}]$,

$$\boldsymbol{m\left(\frac{n-3}{2(n-2)}r + \frac{n-1}{2(n-2)}y\right) = \frac{n-3}{2(n-2)}m(r) + \frac{n-1}{2(n-2)}m(y).} \tag{6}$$

Figure 6 illustrates the step. Let $r \in [0, \frac{s}{n}[$. Since $r < \frac{s}{n}$, then $r < \frac{2s+(n-3)r}{3(n-1)}$. Let $y \in]r, \frac{2s+(n-3)r}{3(n-1)}]$ and $z \equiv \frac{(n-1)y-(n-3)r}{2}$. Since $y > r$, then $z > r$. Let $c^* \in \mathcal{C}(s)$ be the vector defined as follows: for each $j \leq n-3$, $c^*_j \equiv r$, for each $n-2 \leq j \leq n-1$, $c^*_j \equiv z$, and $c^*_n \equiv s-(n-3)r-2z$. Since $c^*_n = s-(n-1)y$ and $y \in]r, \frac{2s+(n-3)r}{3(n-1)}]$, then $c^*_n \geq z$. Thus, $c^*_1 \leq \cdots \leq c^*_n$. By Step 2,

$$S_{n-2}(c^*, E) = E-(n-3)m(r)-2\left[E-(n-1)m\left(\frac{s-\frac{z+s-(n-3)r-2z}{2}}{n-1}\right)\right]. \tag{7}$$

Since $\sum_{i \in N} S_i(c^*, E) = E$, $\min c^* = r$ and $\max c^* = c^*_n$, then by (4),

$$S_{n-2}(c^*, E) = E-(n-3)m(r)-S_{n-1}(c^*, E)-\left[E-(n-1)m\left(\frac{s-(s-(n-3)r-2z)}{n-1}\right)\right]. \tag{8}$$

Since $c^*_{n-2} = c^*_{n-1}$ and S satisfies *e-t-e*, then $S_{n-2}(c^*, E) = S_{n-1}(c^*, E)$. Replacing (7) in (8) yields

$$m(r) = \frac{4(n-1)}{n-3}m\left(\frac{s+(n-3)r+z}{2(n-1)}\right) - \frac{n-1}{n-3}m\left(\frac{(n-3)r+2z}{n-1}\right) - \frac{2}{n-3}E. \tag{9}$$

Since $y \leq \frac{2s+(n-3)r}{3(n-1)}$ and $r \leq \frac{s}{n}$, then $y \leq \frac{s}{n}$. Thus,

$$\frac{(n-3)r+z}{n-2} = \frac{(n-3)r+(n-1)y}{2(n-2)} \leq \frac{s}{n}.$$

Thus, by (5),

$$m\left(\frac{s+(n-3)r+z}{2(n-1)}\right) = \frac{E}{2(n-1)} + \frac{n-2}{2(n-1)}m\left(\frac{(n-3)r+z}{n-2}\right). \tag{10}$$

Since $y = \frac{(n-3)r+2z}{n-1}$, then replacing (10) in (9) and rearranging terms yields

$$m\left(\frac{(n-3)r+z}{n-2}\right) = \frac{n-3}{2(n-2)}m(r) + \frac{n-1}{2(n-2)}m(y). \tag{11}$$

Now, since $\frac{(n-3)r+z}{n-2} = \frac{n-3}{2(n-2)}r + \frac{n-1}{2(n-2)}y$, then $m\left(\frac{n-3}{2(n-2)}r + \frac{n-1}{2(n-2)}y\right) = \frac{n-3}{2(n-2)}m(r) + \frac{n-1}{2(n-2)}m(y)$, as asserted.

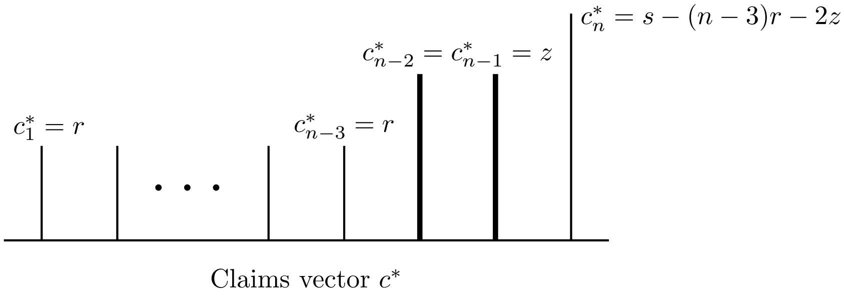


Figure 6: Illustrating Step 3 (Theorem 3). The award to agent $n-2$ can be derived in two ways. One is to consider her as an intermediate claimant (Step 2). The other is to give her the endowment minus what the others (the minimal claimants, the agent with a claim equal to that of agent $n-2$, and the maximal claimant) receive.

Step 4: For each $r \in]0, \frac{2s}{3(n-1)}[$, $m(r) = \frac{3(n-1)}{2s} m\left(\frac{2s}{3(n-1)}\right) r$. Let $L(r) \equiv \{r' \in [0, r] : m(r') = \frac{3(n-1)}{2s} m\left(\frac{2s}{3(n-1)}\right) r'\}$. Since $0 \in L(r)$, then $L(r) \neq \emptyset$. Let $U(r) \equiv \{r' \in [r, \frac{2s}{3(n-1)}] : m(r') = \frac{3(n-1)}{2s} m\left(\frac{2s}{3(n-1)}\right) r'\}$. Since $\frac{2s}{3(n-1)} \in U(r)$, then $U(r) \neq \emptyset$. We prove that $\sup L(r) = \inf U(r)$. Suppose by contradiction that $\sup L(r) < \inf U(r)$. By (P4) –Lemma 2–, $m(\cdot, s, E)$ is continuous. Then, $\sup L(r) \in L(r)$ and $\inf U(r) \in U(r)$. Now, since $\inf U(r) \leq \frac{2s}{3(n-1)}$ and $n > 3$, then $\inf U(r) \leq \frac{2s+(n-3)\sup L(r)}{3(n-1)}$. By (6),

$$m\left(\frac{n-3}{2(n-2)} \sup L(r) + \frac{n-1}{2(n-2)} \inf U(s)\right) = \frac{n-3}{2(n-2)} m(\sup L(r)) + \frac{n-1}{2(n-2)} m(\inf U(s)).$$

Let $v \equiv \frac{n-3}{2(n-2)} \sup L(r) + \frac{n-1}{2(n-2)} \inf U(s)$. Since, $\sup L(r) \in L(r)$ and $\inf U(r) \in U(r)$ then,

$$m(v) = \frac{3(n-1)}{2s} m\left(\frac{2s}{3(n-1)}\right) v.$$

Now, since $v \in [0, \frac{2s}{3(n-1)}]$, then $v \in L(r) \cup U(r)$. This is a contradiction, because $\sup L(r) < v < \inf U(r)$.

Step 5: For each $r \in [0, \frac{s}{n}]$, $m(r) = \frac{r}{s} E$. Let $r \in [0, \frac{s}{n}]$. Since $m(0) = 0$, then by (5), $m\left(\frac{s}{2(n-1)}\right) = \frac{E}{s} \frac{s}{2(n-1)}$. Now, since $\frac{s}{2(n-1)} \leq \frac{2s}{3(n-1)}$, then $m\left(\frac{s}{2(n-1)}\right) = \frac{3(n-1)}{2s} m\left(\frac{2s}{3(n-1)}\right) \frac{s}{2(n-1)}$. Thus, $m\left(\frac{2s}{3(n-1)}\right) = \frac{2}{3(n-1)} E$. By Step 4, for each $r \in [0, \frac{2s}{3(n-1)}]$, $m(r) = \frac{r}{s} E$. Now, let $\{b_t\}_{t \in \mathbb{N}}$ be the sequence defined as follows: for each $t \in \mathbb{N}$,

$$b_t \equiv \frac{1}{n}\left[1 - \left(\frac{n-2}{2(n-1)}\right)^t\right].$$

We prove that for each $t \in \mathbb{N}$ and each $r \in [0, b_t s]$, $m(r) = \frac{r}{s} E$. We prove this by induction on t. Since $b_1 s = \frac{s}{2(n-1)}$, then for each $r \in [0, b_1 s]$, $m(r) = \frac{r}{s} E$. Let $t \geq 1$ and suppose that for each $r \in [0, b_t s]$, $m(r) = \frac{r}{s} E$. We prove that for each $r \in [0, b_{t+1} s]$, $m(r) = \frac{r}{s} E$. Let $r \in [b_t s, b_{t+1} s]$. Since $b_1 s = \frac{s}{2(n-1)} \leq r \leq \frac{s}{n}$, then $0 \leq \frac{2(n-1)r - s}{n-2} \leq \frac{s}{n}$. Thus, by (5),

$$m(r) = \frac{E}{2(n-1)} + \frac{n-2}{2(n-1)} m\left(\frac{2(n-1)r - s}{n-2}\right).$$

Since $r \leq b_{t+1} s$, then $\frac{2(n-1)r - s}{n-2} \leq b_t s$. By the induction hypothesis, $m\left(\frac{2(n-1)r-s}{n-2}\right) = \frac{2(n-1)r-s}{s(n-2)} E$. Replacing $m\left(\frac{2(n-1)r-s}{n-2}\right)$ in the expression above yields that $m(r) = \frac{r}{s} E$. Now, since $b_t s \underset{t \to \infty}{\longrightarrow} \frac{s}{n}$, then for each $r \in [0, \frac{s}{n}[$, $m(r) = \frac{r}{s} E$. Since S satisfies *e-t-e*, then $m(\frac{s}{n}) = \frac{E}{n}$. Thus, for each $r \in [0, \frac{s}{n}]$, $m(r) = \frac{r}{s} E$.

Step 6: For each $(c, E) \in \mathcal{C}$ such that $\bar{c} > 0$, and each $i \in N$, $S_i(c, E) = \frac{c_i}{\bar{c}} E$.

Assume w.l.o.g. that $c_1 \leq \cdots \leq c_n$. By (4) and Step 5, $S_1(c, E) = \frac{c_1}{\bar{c}}E$ and $S_n(c, E) = \frac{c_n}{\bar{c}}E$. Let $i \in N$ be such that $1 < i < n$. By Steps 2 and 5, $S_i(c, E) = \frac{c_i}{\bar{c}}E$. $\square$

Let us remark that the difference behind the results between the cases $n = 3$ and $n > 3$ is mainly due to Step 3 in the proof. Obviously, the functional equation described in Step 3 has no bite for the three-agent case. This functional equation provides completely different restrictions on m from the ones implied by P3 alone.

8 Discussion

In this section, we discuss a way of strengthening our results.[14]

Our main results hold even on domains of problems smaller than $\mathcal{C}$, i.e., the domains of claims problems where the total claim and the endowment are fixed. Let $(s, E) \in \mathbb{R}^2_+$ be such that $E \leq s$. The **small sub-domain of claims problems associated to $\boldsymbol{s}$ and $\boldsymbol{E}$** is the set $\{(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+ : \bar{c} = s\}$.

In the two-agent and three-agent cases, Theorems 1 and 2 hold if claims problems are restricted to a domain composed of an arbitrary union of *small sub-domains*. Here, the notion of continuity of rules is relative to the topology of the domain. Thus, the notion of continuity of minimal award functions has to be defined accordingly in the characterization.

For more than three agents, Theorem 3 holds if claims problems are restricted to a domain composed of an arbitrary union of small domains. So our characterization of the proportional rule can be strengthened by reducing the domain of problems in which the axioms apply.

A Appendix

A.1 Two-agent case

Proof of Theorem 1. $(\Rightarrow)$ Let S be a rule that satisfies the properties in the theorem. By Lemma 2, there is a continuous $m : Y_3 \to \mathbb{R}_+$ satisfying P1, P2, and P3 such that for each $(c, E) \in \mathcal{C}$ each $i \in N$,

$$S_i(c, E) = \begin{cases} m(c_i, \overline{c}, E) & \text{if } c_i = \min c < \max c, \\ E - m(\overline{c} - c_i, \overline{c}, E) & \text{if } c_i = \max c. \end{cases}$$

$(\Leftarrow)$ Let $m : Y_2 \to \mathbb{R}_+$ be a continuous function that satisfies P1 and P2. Let S^m be the rule defined as follows: for each $(c, E) \in \mathcal{C}$ and each $i \in N$,

$$S^m_i(c, E) = \begin{cases} m(c_i, \overline{c}, E) & \text{if } c_i = \min c < \max c, \\ E - m(\overline{c} - c_i, \overline{c}, E) & \text{if } c_i = \max c. \end{cases}$$

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[14] See Ju et al. (2007) for related characterizations on "rich" domains.

We prove that S^m is a well-defined *continuous* rule that satisfies the *core axioms*. Let $(c, E) \in \mathcal{C}$.

- $\boldsymbol{0 \leq S^m(c,E) \leq c}$**.** Let $i \in N$. We prove that $0 \leq S_i^m(c, E) \leq c_i$. There are two cases.

Case 1: $c_i = \min c < \max c$. Since $m(\cdot, \bar{c}, E)$ is non-decreasing and $m(0, \bar{c}, E) = 0$, then $0 \leq M_i^m(c, E)$. Also, by P2, $m(c_i, \bar{c}, E) - m(0, \bar{c}, E) \leq c_i - 0$. Thus, $M_i^m(c, E) \leq c_i$.

Case 2: $c_i = \max c$. Then, $\bar{c} - c_i \leq \bar{c}/2$. Since $m(\cdot, \bar{c}, E)$ is non-decreasing, $m(\bar{c} - c_i, \bar{c}, E) \leq m(\bar{c}/2, \bar{c}, E) = E/2$. Thus, $S_i^m(c, E) = E - m(\bar{c} - c_i, \bar{c}, E) \geq E - E/2 \geq 0$. Now,

$$S_i^m(c, E) = E/2 + [E/2 - m(\bar{c} - c_i, \bar{c}, E)] = E/2 + [m\left(\bar{c}/2, \bar{c}, E\right) - m(\bar{c} - c_i, \bar{c}, E)].$$

Recall that $E \leq \bar{c}$. By P2, $S_i^m(c, E) \leq \bar{c}/2 + [\bar{c}/2 - (\bar{c} - c_i)] = c_i$.

- $\boldsymbol{\sum_{i \in N} S_i^m(c,E) = E}$**.** There are two cases.

Case 1: For each $i \in N$, $c_i = \bar{c}/2$. Since $m\left(\bar{c}/2, \bar{c}, E\right) = E/2$, then

$$\sum_{k \in N} S_k^m(c, E) = 2\left[E - m\left(\bar{c}/2, \bar{c}, E\right)\right] = E.$$

Case 2: There is $i \in N$, such that $c_i = \min c < \max c$. Then,

$$\sum_{k \in N} S_k^m(c, E) = m(\min c, \bar{c}, E) + E - m(\bar{c} - \max c, \bar{c}, E) = E.$$

- ***Continuity.*** Assume $N \equiv \{1, 2\}$. Let $\{(c^k, E^k)\}_{k \in \mathbb{N}}$ be a convergent sequence of problems. Let $(c, E) \equiv \lim_{k \to \infty}(c^k, E^k)$. We prove that $S^m(c^k, E^k) \underset{k \to \infty}{\longrightarrow} S^m(c, E)$. Assume w.l.o.g. that $c_1 \leq c_2$. Note that m is continuous. There are two cases.

Case 1: $c_1 = c_2$. Then, for each $k \in \mathbb{N}$,

$$S_1^m(c^k, E^k) = \begin{cases} m(\min c^k, \bar{c}^k, E^k) & \text{if } c_1^k = \min c^k < \max c^k \\ E^k - m\left(\bar{c}^k - \max c^k, \bar{c}^k, E^k\right) & \text{if } c_1^k = \max c^k. \end{cases}$$

Since $\min c^k \underset{k \to \infty}{\longrightarrow} \bar{c}/2$ and $\max c^k \underset{k \to \infty}{\longrightarrow} \bar{c}/2$, then $m(\min c^k, \bar{c}^k, E^k) \underset{k \to \infty}{\longrightarrow} E/2$, and $m\left(\bar{c}^k - \max c^k, \bar{c}^k, E^k\right) \underset{k \to \infty}{\longrightarrow} E/2$. Thus, $S_1^m(c^k, E^k) \underset{k \to \infty}{\longrightarrow} E/2 = S_1^m(c, E)$. A symmetric argument shows that $S_2^m(c^k, E^k) \underset{k \to \infty}{\longrightarrow} S_2^m(c, E)$.

Case 2: $c_1 < c_2$. Since $c^k \underset{k \to \infty}{\longrightarrow} c$, then there is $K \in \mathbb{N}$ such that for each $k \geq K$, $c_1^k < c_2^k$ and thus, $S_1^m(c^k, E^k) = m(c_1^k, \bar{c}^k, E^k)$ and $S_2^m(c^k, E^k) = E^k - m(\bar{c}^k - c_2^k, \bar{c}^k, E^k)$. Thus, $S_1^m(c^k, E^k) \underset{k \to \infty}{\longrightarrow} m(c_1, \bar{c}, E) = S_1^m(c, E)$ and $S_2^m(c^k, E^k) \underset{k \to \infty}{\longrightarrow} E - m(\bar{c} - c_2, \bar{c}, E) = S_2^m(c, E)$.

- ***o-p-g* and *o-p-l.*** Assume $N \equiv \{1, 2\}$. Let $(c, E) \in \mathcal{C}$ be such that $c_1 \leq c_2$. We prove that $S_1^m(c, E) \leq S_2^m(c, E)$ and $c_1 - S_1^m(c, E) \leq c_2 - S_2^m(c, E)$. There are two cases.

Case 1: $c_1 = c_2$. Then, $c_1 = c_2 = \max c$. Thus, $S_1^m(c, E) = S_2^m(c, E)$ and $c_1 - S_1^m(c, E) = c_2 - S_2^m(c, E)$.

Case 2: $c_1 < c_2$. Since $m\left(\bar{c}/2, \bar{c}, E\right) = E/2$ and m is non-decreasing, then $m(c_1, \bar{c}, E) \leq E/2$. Equivalently, $m(c_1, \bar{c}, E) \leq E - m(\bar{c} - c_2, \bar{c}, E)$, i.e., $S_1^m(c, E) \leq S_2^m(c, E)$. Also by P2, $m\left(\bar{c}/2, \bar{c}, E\right) - m(c_1, \bar{c}, E) \leq \bar{c}/2 - c_1$. Thus, $E - 2m(c_1, \bar{c}, E) \leq c_2 - c_1$. Equivalently, $c_1 - m(c_1, \bar{c}, E) \leq c_2 - [E - m(\bar{c} - c_2, \bar{c}, E)]$, i.e., $c_1 - S_1^m(c, E) \leq c_2 - S_2^m(c, E)$.

- ***i-p-g.*** Let $s \in \mathbb{R}_+$ and $\{(c, E), (c', E)\} \subset \mathcal{C}(s)$ be such that $c' \succeq_L c$. We prove that $S^m(c', E) \succeq_L S^m(c, E)$. It is enough to show that $S^m_{[1]}(c', E) \geq S^m_{[1]}(c, E)$. Note that $\overline{c} = \overline{c}' = s$. There are three cases.

Case 1: $\min c < \bar{c}/2$ and $\min c' < \bar{c}'/2$. Since S^m satisfies *o-p-g*, $S^m_{[1]}(c', E) = m(\min c', \overline{c}, E)$ and $S^m_{[1]}(c, E) = m(\min c, \overline{c}, E)$. Since $c' \succeq_L c$, $\min c' \geq \min c$. Since m is non-decreasing, $m(\min c', \overline{c}, E) \geq m(\min c, \overline{c}, E)$. Thus, $S^m_{[1]}(c', E) \geq S^m_{[1]}(c, E)$.

Case 2: $\min c < \bar{c}/2$ and $\min c' = \bar{c}'/2$. Since S^m satisfies *o-p-g*, $S^m_{[1]}(c', E) = E/2$ and $S^m_{[1]}(c, E) = m(\min c, \overline{c}, E)$. Since m is non-decreasing, $m(\min c, \overline{c}, E) \leq m(\bar{c}/2, \overline{c}, E) = E/2$. Thus, $S^m_{[1]}(c', E) \geq S^m_{[1]}(c, E)$.

Case 3: $\min c = \bar{c}/2$. Since $c' \succeq_L c$, $\min c' = \bar{c}'/2$. Then, $S^m_{[1]}(c', E) = S^m_{[1]}(c, E) = E/2$. Thus, $S^m_{[1]}(c', E) \geq S^m_{[1]}(c, E)$.

- ***i-p-l.*** Let $s \in \mathbb{R}_+$ and $\{(c, E), (c', E)\} \subset \mathcal{C}(s)$ be such that $c' \succeq_L c$. We prove that $c' - S^m(c', E) \succeq_L c - S^m(c, E)$. It is enough to show that $[c' - S^m(c', E)]_{[1]} \geq [c - S^m(c, E)]_{[1]}$. Note that $\overline{c} = \overline{c}' = s$. There are three cases.

Case 1: $\min c < \bar{c}/2$ and $\min c' < \bar{c}'/2$. Since S^m satisfies *o-p-l*, $[c' - S^m(c', E)]_{[1]} = \min c' - m(\min c', \overline{c}, E)$ and $[c - S^m(c, E)]_{[1]} = \min c - m(\min c, \overline{c}, E)$. Since $c' \succeq_L c$, $\min c' \geq \min c$. Then by P2, $m(\min c', \overline{c}, E) - m(\min c, \overline{c}, E) \leq \min c' - \min c$. Thus, $[c' - S^m(c', E)]_{[1]} \geq [c - S^m(c, E)]_{[1]}$.

Case 2: $\min c < \bar{c}/2$ and $\min c' = \bar{c}'/2$. Since S^m satisfies *o-p-l*, $[c' - S^m(c', E)]_{[1]} = \bar{c}'/2 - E/2$ and $[c - S^m(c, E)]_{[1]} = \min c - m(\min c, \overline{c}, E)$. By P2, $m(\bar{c}/2, \overline{c}, E) - m(\min c, \overline{c}, E) \leq \bar{c}/2 - \min c$, i.e., $E/2 - m(\min c, \overline{c}, E) \leq \bar{c}'/2 - \min c$. Thus, $[c' - S^m(c', E)]_{[1]} \geq [c - S^m(c, E)]_{[1]}$.

Case 3: $\min c = \bar{c}/2$. Since $c' \succeq_L c$, $\min c' = \bar{c}'/2$. Then, $[c' - S^m(c', E)]_{[1]} = [c - S^m(c, E)]_{[1]} = \bar{c}/2 - E/2$. Thus, $S^m_{[1]}(c', E) \geq S^m_{[1]}(c, E)$. □

A.2 Three-agent case

Proof of Theorem 2. (⇒) Let S be a rule that satisfies the properties in the theorem. By Lemma 2, there is a continuous $m : Y_3 \to \mathbb{R}_+$ satisfying P1, P2, and P3 such that for each $(c, E) \in \mathcal{C}$ each $i \in N$,

$$S_i(c, E) = \begin{cases} m(c_i, \overline{c}, E) & \text{if } c_i = \min c < \max c, \\ E - 2m(\frac{\overline{c} - c_i}{2}, \overline{c}, E) & \text{if } c_i = \max c. \end{cases}$$

Suppose now that $i \in N$ is such that $\min c < c_i < \max c$. Since $\sum_{j \in N} S_j(c, E) = E$, then $S_i(c, E) = E - m(\min c, \overline{c}, E) - [E - 2m(\frac{\overline{c} - \max c}{2}, \overline{c}, E)]$.

(⇐) Let $m : Y_3 \to \mathbb{R}_+$ be a continuous function that satisfies P1, P2, and P3. Let S^m be the rule defined as follows: for each $(c, E) \in \mathcal{C}$ and each $i \in N$,

$$S^m_i(c, E) = \begin{cases} m(c_i, \overline{c}, E) & \text{if } c_i = \min c < \max c, \\ E - m(\min c, \overline{c}, E) - [E - 2m(\frac{\overline{c} - \max c}{2}, \overline{c}, E)] & \text{if } \min c < c_i < \max c, \\ E - 2m(\frac{\overline{c} - c_i}{2}, \overline{c}, E) & \text{if } c_i = \max c. \end{cases}$$

We prove that S^m is a well-defined *continuous* rule that satisfies the *core axioms*. Let $(c, E) \in \mathcal{C}$.

- $\boldsymbol{0 \leq S^m(c, E) \leq c}$**.** Let $i \in N$. We prove that $0 \leq S^m_i(c, E) \leq c_i$. There are three cases.

Case 1: $c_i = \min c < \max c$. Since $m(\cdot, \bar{c}, E)$ is non-decreasing and $m(0, \bar{c}, E) = 0$, then $0 \leq m(c_i, \bar{c}, E)$. Since m satisfies P2, $m(c_i, \bar{c}, E) - m(0, \bar{c}, E) \leq c_i - 0$. Thus, $m(c_i, \bar{c}, E) \leq c_i$.

Case 2: $\min c < c_i < \max c$. Then, $\bar{c} = \min c + c_i + \max c$. Thus,

$$S_i^m(c, E) = m\left(\frac{\min c + c_i}{2}, \bar{c}, E\right) + \left[m\left(\frac{\min c + c_i}{2}, \bar{c}, E\right) - m\left(\min c, \bar{c}, E\right)\right].$$

Since $m(\cdot, \bar{c}, E)$ is non-decreasing and $m(0, \bar{c}, E) = 0$, then $0 \leq S_i^m(c, E)$. Since m satisfies P2, $S_i^m(c, E) \leq \frac{\min c + c_i}{2} + \frac{\min c + c_i}{2} - \min c = c_i$.

Case 3: $c_i = \max c$. Then, $\frac{\bar{c} - c_i}{2} \leq \frac{\bar{c}}{3}$. Since $m(\cdot, \bar{c}, E)$ is non-decreasing, $m(\frac{\bar{c} - c_i}{2}, \bar{c}, E) \leq m(\frac{\bar{c}}{3}, \bar{c}, E) = \frac{E}{3}$. Thus, $S_i^m(c, E) = E - 2m\left(\frac{\bar{c} - c_i}{2}, \bar{c}, E\right) \geq E - \frac{2}{3}E \geq 0$. Now,

$$S_i^m(c, E) = \frac{E}{3} + 2\left[\frac{E}{3} - m\left(\frac{\bar{c} - c_i}{2}, \bar{c}, E\right)\right] = \frac{E}{3} + 2\left[m\left(\frac{\bar{c}}{3}, \bar{c}, E\right) - m\left(\frac{\bar{c} - c_i}{2}, \bar{c}, E\right)\right].$$

Recall that $E \leq \bar{c}$. Since m satisfies P2, then $S_i^m(c, E) \leq \frac{\bar{c}}{3} + 2\left[\frac{\bar{c}}{3} - \frac{\bar{c} - c_i}{2}\right] = c_i$.

- $\boldsymbol{\sum_{i \in N} S_i^m(c, E) = E}$**.** There are four cases.

Case 1: There are $\{i, j\} \subset N$ such that $c_i = c_j = \min c < \max c$. Then, $c_i = c_j = \frac{\bar{c} - \max c}{2}$. Thus, $\sum_{k \in N} S_k^m(c, E) = m(\min c, \bar{c}, E) + m(\min c, \bar{c}, E) + [E - 2m(\min c, \bar{c}, E)] = E$.

Case 2: There are $\{i, j\} \subset N$ such that $c_i = c_j = \max c > \min c$. Then, $\max c \in \left[\frac{\bar{c}}{3}, \frac{\bar{c}}{2}\right]$. Thus, $\frac{\bar{c} - \max c}{2} \in \left[\frac{\bar{c}}{4}, \frac{\bar{c}}{3}\right]$. Since $4\frac{\bar{c} - \max c}{2} - \bar{c} = \bar{c} - 2\max c = \min c$ and m satisfies P3, then $m\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right) = \frac{E}{4} + \frac{1}{4}m(\min c, \bar{c}, E)$.[15] Thus,

$$\sum_{k \in N} S_k^m(c, E) = 2\left[E - 2m\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right)\right] + m(\min c, \bar{c}, E) = E.$$

Case 3: For each $i \in N$, $c_i = \bar{c}/3$. Since $m\left(\bar{c}/3, \bar{c}, E\right) = E/3$, then

$$\sum_{k \in N} S_k^m(c, E) = 3\left[E - 2m\left(\bar{c}/3, \bar{c}, E\right)\right] = E.$$

Case 4: There is $i \in N$, such that $\min c < c_i < \max c$. Then,

$$\sum_{k \in N} S_k^m(c, E) = \left\{\begin{array}{l} m(\min c, \bar{c}, E) + \\ E - m(\min c, c, E) - [E - 2m(\frac{\bar{c} - \max c}{2}, \bar{c}, E)] + \\ E - 2m(\frac{\bar{c} - \max c}{2}, \bar{c}, E) \end{array}\right\} = E.$$

- ***Continuity.*** Assume $N \equiv \{1, 2, 3\}$. Let $\{(c^k, E^k)\}_{k \in \mathbb{N}}$ be a convergent sequence of problems. Let $(c, E) \equiv \lim_{k \to \infty}(c^k, E^k)$. We prove that $S^m(c^k, E^k) \underset{k \to \infty}{\longrightarrow} S^m(c, E)$. Assume w.l.o.g. that $c_1 \leq c_2 \leq c_3$. Note that m is continuous. There are four cases.

Case 1: $c_1 = c_2 < c_3$. Since $c^k \underset{k \to \infty}{\longrightarrow} c$, then there is $K \in \mathbb{N}$ such that for each $k \geq K$, $c_1^k < c_3^k$

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[15] The function, $r' \in [s/4, s/3] \mapsto 4r' - s$, is a bijection from $[s/4, s/3]$ to $[0, s/3]$. Thus, the statement in P3 is equivalent to: for each $(r, s, E) \in Y_3$ such that $r \in [s/4, s/3]$, $m(r, s, E) = \frac{E}{4} + \frac{1}{4}m(4r - s, s, E)$.

and $c_2^k < c_3^k$, and thus

$$S_1^m(c^k, E^k) = \begin{cases} m(\min c^k, \bar{c}^k, E^k) & \text{if } c_1^k = \min c^k < \max c^k \\ 2m(\frac{\bar{c}^k - \max c^k}{2}, \bar{c}^k, E^k) - m(\min c^k, \bar{c}^k, E^k) & \text{if } \min c^k < c_1^k < \max c^k, \end{cases}$$

and $S_3^m(c^k, E^k) = E^k - 2m(\frac{\bar{c}^k - \max c^k}{2}, \bar{c}^k, E^k)$. Thus, $S_3^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} E - 2m(\frac{\bar{c} - \max c}{2}, \bar{c}, E) = S_3^m(c, E)$. Now, since $\frac{\bar{c}^k - \max c^k}{2} \underset{k\to\infty}{\longrightarrow} \min c$, then

$$S_1^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} m(\min c, \bar{c}, E) = S_1^m(c, E).$$

Since $S_2^m(c^k, E^k) = E^k - S_1^m(c^k, E^k) - S_3^m(c^k, E^k)$, then $S_2^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} S_2^m(c, E)$.

Case 2: $c_1 < c_2 = c_3$. Since $c^k \underset{k\to\infty}{\longrightarrow} c$, then there is $K \in \mathbb{N}$ such that for each $k \geq K$, $c_1^k < c_2^k$ and $c_1^k < c_3^k$, and thus

$$S_2^m(c^k, E^k) = \begin{cases} 2m(\frac{\bar{c}^k - \max c^k}{2}, \bar{c}^k, E^k) - m(\min c^k, \bar{c}^k, E^k) & \text{if } \min c^k < c_2^k < \max c^k \\ E^k - 2m(\frac{\bar{c}^k - \max c^k}{2}, \bar{c}^k, E^k) & \text{if } c_2^k = \max c^k, \end{cases}$$

and $S_1^m(c^k, E^k) = m(\min c^k, \bar{c}^k, E^k)$. Thus, $S_1^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} S_1^m(c, E)$. Now, since $\frac{\bar{c} - \max c}{2} \in [\frac{\bar{c}}{4}, \frac{\bar{c}}{3}]$ and $\min c = 4\frac{\bar{c} - \max c}{2} - \bar{c}$, then by P3,

$$2m\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right) - m(\min c, \bar{c}, E) = E - 2m\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right).$$

Thus, $S_2^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} S_2^m(c, E)$. Since $S_3^m(c^k, E^k) = E^k - S_1^m(c^k, E^k) - S_2^m(c^k, E^k)$, then $S_3^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} S_3^m(c, E)$.

Case 3: $c_1 = c_2 = c_3$. Then, for each $k \in \mathbb{N}$,

$$S_1^m(c^k, E^k) = \begin{cases} m(\min c^k, \bar{c}^k, E^k) & \text{if } c_1^k = \min c^k < \max c^k \\ 2m(\frac{\bar{c}^k - \max c^k}{2}, \bar{c}^k, E^k) - m(\min c^k, \bar{c}^k, E^k) & \text{if } \min c^k < c_1^k < \max c^k \\ E^k - 2m\left(\frac{\bar{c}^k - \max c^k}{2}, \bar{c}^k, E^k\right) & \text{if } c_1^k = \max c^k. \end{cases}$$

Since $\min c^k \underset{k\to\infty}{\longrightarrow} \frac{\bar{c}}{3}$ and $\max c^k \underset{k\to\infty}{\longrightarrow} \frac{\bar{c}}{3}$, then $m(\min c^k, \bar{c}^k, E^k) \underset{k\to\infty}{\longrightarrow} \frac{E}{3}$, and $m\left(\frac{\bar{c}^k - \max c^k}{2}, \bar{c}^k, E^k\right) \underset{k\to\infty}{\longrightarrow} \frac{E}{3}$. Thus, $S_1^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} \frac{E}{3} = S_1^m(c, E)$. A symmetric argument shows that $S_2^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} S_2^m(c, E)$ and $S_3^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} S_3^m(c, E)$.

Case 4: $c_1 < c_2 < c_3$. Since $c^k \underset{k\to\infty}{\longrightarrow} c$, then there is $K \in \mathbb{N}$ such that for each $k \geq K$, $c_1^k < c_2^k < c_3^k$ and thus,

$$S_1^m(c^k, E^k) = m(c_1^k, \bar{c}^k, E^k) \text{ and } S_3^m(c^k, E^k) = E^k - 2m\left(\frac{\bar{c}^k - c_3^k}{2}, \bar{c}^k, E^k\right).$$

Thus, $S_1^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} m(c_1, \bar{c}, E) = S_1^m(c, E)$ and $S_3^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} E - 2m(\frac{\bar{c}-c_3}{2}, \bar{c}, E) = S_3^m(c, E)$. Since $S_2^m(c^k, E^k) = E^k - S_1^m(c^k, E^k) - S_3^m(c^k, E^k)$, then $S_2^m(c^k, E^k) \underset{k\to\infty}{\longrightarrow} S_2^m(c, E)$.

- ***o-p-g* and *o-p-l*.** Let $(c, E) \in \mathcal{C}$. Assume $\{1, 2\} \subset N$ are such that $c_1 \leq c_2$. We prove that $S_1^m(c, E) \leq S_2^m(c, E)$ and $c_1 - S_1^m(c, E) \leq c_2 - S_2^m(c, E)$. There are five cases.

Case 1: $c_1 = c_2 = \min c$ and $\min c \neq \max c$. Then, $S_1^m(c, E) = S_2^m(c, E)$ and $c_1 - S_1^m(c, E) = c_2 - S_2^m(c, E)$.

Case 2: $c_1 = \min c$ and $\min c < c_2 < \max c$. Since m is non-decreasing, $m(c_1, \overline{c}, E) \leq m(\frac{c_1+c_2}{2}, \overline{c}, E)$. Thus,

$$m(c_1, \overline{c}, E) \leq E - m(c_1, \overline{c}, E) - \left[E - 2m\left(\frac{c_1 + c_2}{2}, \bar{c}, E\right)\right],$$

i.e., $S_1^m(c, E) \leq S_2^m(c, E)$ (note that $\frac{\overline{c}-\max c}{2} = \frac{c_1+c_2}{2}$). By P2 (take $r = c_1$ and $r' = \frac{c_1+c_2}{2}$), $m(\frac{c_1+c_2}{2}, \overline{c}, E) - m(c_1, \overline{c}, E) \leq \frac{c_1+c_2}{2} - c_1$. Thus,

$$c_1 - m(c_1, \overline{c}, E) \leq c_2 - 2m\left(\frac{c_1 + c_2}{2}, \overline{c}, E\right) + m(c_1, \overline{c}, E),$$

i.e., $c_1 - S_1^m(c, E) \leq c_2 - S_2^m(c, E)$.

Case 3: $\min c < c_1 < \max c$ and $c_2 = \max c$. There are two subcases.

Case 3 (a): If $c_2 \leq \frac{\overline{c}}{2}$. Then, $\frac{\overline{c}-c_2}{2} \in [\frac{\overline{c}}{4}, \frac{\overline{c}}{3}]$. By P3, $m(\frac{\overline{c}-c_2}{2}, \overline{c}, E) = \frac{E}{4} + \frac{1}{4}m(\overline{c} - 2c_2, \overline{c}, E)$. Since m is non-decreasing, $m(\overline{c} - 2c_2, \overline{c}, E) = m(\min c + (c_1 - c_2), \overline{c}, E) \leq m(\min c, \overline{c}, E)$. Thus, $m(\frac{\overline{c}-c_2}{2}, \overline{c}, E) \leq \frac{E}{4} + \frac{1}{4}m(\min c, \overline{c}, E)$, i.e., $S_1^m(c, E) \leq S_2^m(c, E)$. By P2 (take $r = \min c + (c_1 - c_2)$ and $r' = \min c$), $m(\min c, \overline{c}, E) - m(\min c + (c_1 - c_2), \overline{c}, E) \leq c_2 - c_1$. Thus, $c_1 - m(\overline{c} - 2c_2, \overline{c}, E) \leq c_2 - m(\min c, \overline{c}, E)$. Hence $c_1 - 4m(\frac{\overline{c}-c_2}{2}, \overline{c}, E) + E \leq c_2 - m(\min c, \overline{c}, E)$, i.e., $c_1 - S_1^m(c, E) \leq c_2 - S_2^m(c, E)$.

Case 3 (b): If $c_2 > \frac{\overline{c}}{2}$. Then, $\frac{\overline{c}-c_2}{2} \in [0, \frac{\overline{c}}{4}[$. Since $m(\frac{\overline{c}}{4}, \overline{c}, E) = \frac{E}{4}$ and m is non-decreasing, $m(\frac{\overline{c}-c_2}{2}, \overline{c}, E) \leq \frac{E}{4}$. Since $m(\min c, \overline{c}, E) \geq 0$, we have $m(\frac{\overline{c}-c_2}{2}, \overline{c}, E) \leq \frac{E}{4} + \frac{1}{4}m(\min c, \overline{c}, E)$, i.e., $S_1^m(c, E) \leq S_2^m(c, E)$. By P2 (take $r = \frac{\overline{c}-c_2}{2}$ and $r' = \frac{\overline{c}}{4}$), $m(\frac{\overline{c}}{4}, \overline{c}, E) - m(\frac{\overline{c}-c_2}{2}, \overline{c}, E) \leq \frac{\overline{c}}{4} - \frac{\overline{c}-c_2}{2}$. Since $\min c - m(\min c, \overline{c}, E) \geq 0$, we have

$$m\left(\frac{\overline{c}}{4}, \overline{c}, E\right) - m\left(\frac{\overline{c} - c_2}{2}, \overline{c}, E\right) \leq \frac{\overline{c}}{4} - \frac{\overline{c} - c_2}{2} + \frac{\min c - m(\min c, \overline{c}, E)}{4},$$

i.e., $c_1 - S_1^m(c, E) \leq c_2 - S_2^m(c, E)$ (note that $m(\frac{\overline{c}}{4}, \overline{c}, E) = \frac{E}{4}$ and $\frac{\overline{c}}{4} - \frac{\overline{c}-c_2}{2} + \frac{\min c}{4} = \frac{c_2-c_1}{4}$).

Case 4: $c_1 = \min c$ and $c_2 = \max c$. It follows from Cases 2 and 3.

Case 5: $c_1 = c_2 = \max c$. Then, $S_1^m(c, E) = S_2^m(c, E)$ and $c_1 - S_1^m(c, E) = c_2 - S_2^m(c, E)$.

- ***i-p-g*.** Let $s \in \mathbb{R}_+$ and $\{(c, E), (c', E)\} \subset \mathcal{C}(s)$ be such that $c' \succeq_L c$. We prove that $S^m(c', E) \succeq_L S^m(c, E)$. It is enough to show that $S_{[1]}^m(c', E) \geq S_{[1]}^m(c, E)$ and $S_{[3]}^m(c', E) \leq S_{[3]}^m(c, E)$. Note that $\overline{c} = \overline{c}' = s$. There are three cases.

Case 1: $\min c < \overline{c}/3$ and $\min c' < \overline{c}'/3$. Since S^m satisfies *o-p-g*,

$$S_{[1]}^m(c', E) = m(\min c', \overline{c}, E) \quad \text{and} \quad S_{[1]}^m(c, E) = m(\min c, \overline{c}, E).$$

Since $c' \succeq_L c$, $\min c' \geq \min c$. Since m is non-decreasing, $m(\min c', \overline{c}, E) \geq m(\min c, \overline{c}, E)$. Thus, $S^m_{[1]}(c', E) \geq S^m_{[1]}(c, E)$. Similarly, by *o-p-g*,

$$S^m_{[3]}(c', E) = E - 2m\left(\frac{\overline{c} - \max c'}{2}, \overline{c}, E\right) \quad \text{and} \quad S^m_{[3]}(c, E) = E - 2m\left(\frac{\overline{c} - \max c}{2}, \overline{c}, E\right).$$

Since $c' \succeq_L c$, $\max c' \leq \max c$. Since m is non-decreasing, $m(\frac{\overline{c} - \max c'}{2}, \overline{c}, E) \geq m(\frac{\overline{c} - \max c}{2}, \overline{c}, E)$. Thus, $S^m_{[3]}(c', E) \leq S^m_{[3]}(c, E)$.

Case 2: $\min c < \overline{c}/3$ and $\min c' = \overline{c}'/3$. Since S^m satisfies *o-p-g*, then $S^m_{[1]}(c', E) = E/3$ and $S^m_{[1]}(c, E) = m(\min c, \overline{c}, E)$. Since m is non-decreasing, $m(\min c, \overline{c}, E) \leq m\left(\overline{c}/3, \overline{c}, E\right) = E/3$. Thus, $S^m_{[1]}(c', E) \geq S^m_{[1]}(c, E)$. Similarly, by *o-p-g*,

$$S^m_{[3]}(c', E) = \frac{E}{3} \quad \text{and} \quad S^m_{[3]}(c, E) = E - 2m\left(\frac{\overline{c} - \max c}{2}, \overline{c}, E\right).$$

Since m is non-decreasing, $m(\frac{\overline{c} - \max c}{2}, \overline{c}, E) \leq m(\frac{\overline{c}}{3}, \overline{c}, E) = \frac{E}{3}$. Thus, $S^m_{[3]}(c', E) \leq S^m_{[3]}(c, E)$.

Case 3: $\min c = \overline{c}/3$. Since $c' \succeq_L c$, $\min c' = \overline{c}'/3$. Then,

$$S^m_{[1]}(c', E) = S^m_{[1]}(c, E) = S^m_{[3]}(c', E) = S^m_{[3]}(c, E) = E/3.$$

Thus, $S^m_{[1]}(c', E) \geq S^m_{[1]}(c, E)$ and $S^m_{[3]}(c', E) \leq S^m_{[3]}(c, E)$.

- ***i-p-l.*** Let $s \in \mathbb{R}_+$ and $\{(c, E), (c', E)\} \subset \mathcal{C}(s)$ be such that $c' \succeq_L c$. We prove that $c' - S^m(c', E) \succeq_L c - S^m(c, E)$. It is enough to show that $[c' - S^m(c', E)]_{[1]} \geq [c - S^m(c, E)]_{[1]}$ and $[c' - S^m(c', E)]_{[3]} \leq [c - S^m(c, E)]_{[3]}$. Note that $\overline{c} = \overline{c}' = s$. There are three cases.

Case 1: $\min c < \overline{c}/3$ and $\min c' < \overline{c}'/3$. Since S^m satisfies *o-p-l*,

$$[c' - S^m(c', E)]_{[1]} = \min c' - m(\min c', \overline{c}, E) \quad \text{and} \quad [c - S^m(c, E)]_{[1]} = \min c - m(\min c, \overline{c}, E).$$

Since $c' \succeq_L c$, $\min c' \geq \min c$. By P2 (take $r = \min c$ and $r' = \min c'$), $m(\min c', \overline{c}, E) - m(\min c, \overline{c}, E) \leq \min c' - \min c$. Thus, $[c' - S^m(c', E)]_{[1]} \geq [c - S^m(c, E)]_{[1]}$. Similarly, by *o-p-l*,

$$[c' - S^m(c', E)]_{[3]} = \max c' - E + 2m\left(\frac{\overline{c} - \max c'}{2}, \overline{c}, E\right) \quad \text{and}$$

$$[c - S^m(c, E)]_{[3]} = \max c - E + 2m\left(\frac{\overline{c} - \max c}{2}, \overline{c}, E\right).$$

Since $c' \succeq_L c$, $\max c' \leq \max c$. By P2 (take $r = \frac{\overline{c} - \max c}{2}$ and $r' = \frac{\overline{c} - \max c'}{2}$), $m(\frac{\overline{c} - \max c'}{2}, \overline{c}, E) - m(\frac{\overline{c} - \max c}{2}, \overline{c}, E) \leq \frac{\max c - \max c'}{2}$. Thus, $[c' - S^m(c', E)]_{[3]} \leq [c - S^m(c, E)]_{[3]}$.

Case 2: $\min c < \overline{c}/3$ and $\min c' = \overline{c}'/3$. Since S^m satisfies *o-p-l*,

$$[c' - S^m(c', E)]_{[1]} = \overline{c}'/3 - E/3 \quad \text{and} \quad [c - S^m(c, E)]_{[1]} = \min c - m(\min c, \overline{c}, E).$$

By P2 (take $r = \min c$ and $r' = \overline{c}/3$), $m\left(\overline{c}/3, \overline{c}, E\right) - m(\min c, \overline{c}, E) \leq \overline{c}/3 - \min c$, i.e., $E/3 - m(\min c, \overline{c}, E) \leq \overline{c}'/3 - \min c$. Thus, $[c' - S^m(c', E)]_{[1]} \geq [c - S^m(c, E)]_{[1]}$. Similarly, by *o-p-l*,

$$[c' - S^m(c', E)]_{[3]} = \frac{\bar{c}'}{3} - \frac{E}{3} \quad \text{and} \quad [c - S^m(c, E)]_{[3]} = \max c - E + 2m\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right).$$

By P2 (take $r = \frac{\bar{c}-\max c}{2}$ and $r' = \frac{\bar{c}}{3}$),

$$m\left(\frac{\bar{c}}{3}, \bar{c}, E\right) - m\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right) \leq \frac{\bar{c}}{3} - \frac{\bar{c} - \max c}{2},$$

i.e., $\frac{E}{3} - m\left(\frac{\bar{c}-\max c}{2}, \bar{c}, E\right) \leq \frac{\bar{c}'}{3} - \frac{\bar{c}'-\max c}{2}$. Thus, $[c' - S^m(c', E)]_{[3]} \leq [c - S^m(c, E)]_{[3]}$.

Case 3: $\min c = \bar{c}/3$. Since $c' \succeq_L c$, $\min c' = \bar{c}'/3$. Then,

$$[c' - S^m(c', E)]_{[1]} = [c - S^m(c, E)]_{[1]} = [c' - S^m(c', E)]_{[3]} = [c - S^m(c, E)]_{[3]} = \bar{c}/3 - E/3.$$

Thus, $[c' - S^m(c', E)]_{[1]} \geq [c - S^m(c, E)]_{[1]}$ and $[c' - S^m(c', E)]_{[3]} \leq [c - S^m(c, E)]_{[3]}$. □

Proof of Proposition 1. We first prove that m_f is well-defined. It is enough to show that for each $(r, s, E) \in Y_3$ such that $r \in]0, s/3[$, there is a unique $t \in \{0\} \cup \mathbb{N}$ such that $r \in]a_t s, a_{t+1} s]$; moreover, for such a t, $4^t(r - a_t s) \in]0, s/4]$. Since $\{a_t\}_{t \in \{0\} \cup \mathbb{N}}$ is an increasing sequence such that $a_0 = 0$ and $a_t \xrightarrow[t \to \infty]{} 1/3$, then for each $r \in]0, s/3[$, there is a unique $t \in \{0\} \cup \mathbb{N}$ such that $r \in]a_t s, a_{t+1} s]$. If $t = 0$, then $r = 4^0(r - a_0 s) \in]a_0 s, a_1 s] =]0, s/4]$. Let $T \in \mathbb{N}$. Suppose that for each $t \leq T - 1$, if $r \in]a_t s, a_{t+1} s]$ then $4^t(r - a_t s) \in]0, s/4]$. We prove that for each $r \in]a_T s, a_{T+1} s]$, $4^T(r - a_T s) \in]0, s/4]$. Let $r \in]a_T s, a_{T+1} s]$. Then, $4r - s \in]4a_T s - s, 4a_{T+1} s - s] =]a_{T-1} s, a_T s]$. By the induction hypothesis $4^{T-1}(4r - s - a_{T-1} s) = 4^T(r - a_T s) \in]0, s/4]$.

We now prove the proposition.

($\Rightarrow$) Let $m : Y_3 \to \mathbb{R}_+$ be a continuous function that satisfies P1, P2, and P3. We prove that there is $f \in \mathcal{F}$ such that $m = m_f$. Recall that since m satisfies P3, then for each $(s, E) \in \mathbb{R}^2_+$ such that $s \geq E$ and each $r \in [0, s/3]$,

$$m\left(\frac{s + r}{4}, s, E\right) = \frac{E}{4} + \frac{1}{4} m(r, s, E). \tag{12}$$

We complete the proof in two steps.

Step 1: Identifying initial minimal award function. Let $f : X \to \mathbb{R}_+$ be the restriction of m to X, i.e., $f \equiv m|_X$. We prove that $f \in \mathcal{F}$. Since m is continuous, so is f. Let $(s, E) \in \mathbb{R}^2_+$ be such that $s \geq E$. **(p1)** By property P1, $f(\cdot, s, E) : [0, s/4] \to \mathbb{R}_+$ is a non-decreasing function such that $f(0, s, E) = 0$. **(p2)** By property P2, for each $\{r, r'\} \subset [0, s/4]$ such that $r' \geq r$, $f(r', s, E) - f(r, s, E) \leq r' - r$. **(a1)** Taking $r = 0$ in (12), $m\left(\frac{s}{4}, s, E\right) = \frac{E}{4} + \frac{1}{4} m(0, s, E) = \frac{E}{4}$. Thus, $f \in \mathcal{F}$.

Step 2: $m = m_f$. Let $z \equiv (r, s, E) \in Y_3$. We prove that $m(z) = m_f(z)$. There are three cases.

Case 1: $r = 0$. By P1, $m(0, s, E) = 0$. Thus, $m(z) = m_f(z)$.

Case 2: $r = s/3$. By P1, $m(s/3, s, E) = E/3$. Thus, $m(z) = m_f(z)$.

Case 3: $r \in]0, s/3[$. We assert that for each $t \in \{0\} \cup \mathbb{N}$, if $r \in]a_t s, a_{t+1}s]$, then $m(z) = m_f(z)$. We prove this assertion by induction on t. Since $f = m|_X$, then the assertion is true for $t = 0$. Suppose now that the assertion is true for each $t < T$. Let $r \in]a_T s, a_{T+1}s]$. Since $r \in]s/4, s/3[$ then $4r - s \in]0, s/3[$. Thus, by (12), $m(z) = \frac{E}{4} + \frac{1}{4}m(4r - s, s, E)$. By the induction hypothesis (recall that $4r - s \in]a_{T-1}s, a_T s]$),

$$m(z) = \frac{E}{4} + \frac{1}{4}\left[a_{T-1}E + \frac{1}{4^{T-1}}f(4^{T-1}(4r - s - a_{T-1}s), s, E)\right].$$

Thus, $m(z) = a_T E + \frac{1}{4^T}f(4^T(r - a_T s), s, E)$. Thus, $m(z) = m_f(z)$.

Uniqueness. If $m' : Y_3 \to \mathbb{R}_+$ is a continuous function that satisfies P1, P2, P3, and coincides with m on X, then $m'|_X = f$. The same argument as above shows that $m' = m_f$. Thus, $m = m'$.

($\Leftarrow$) Let $f \in \mathcal{F}$. We prove that m_f is continuous, satisfies P1, P2, P3, and has values in $\mathbb{R}_+$.

• $\boldsymbol{m_f}$ **is continuous.** let $\{z^k \equiv (r^k, s^k, E^k)\}_{k\in\mathbb{N}}$ be a convergent sequence in Y_3. Let $z = (r, s, E) \equiv \lim_{k\to\infty} z^k$. We prove that $m_f(z^k) \underset{k\to\infty}{\longrightarrow} m_f(z)$. There are four cases.

Case 1: $r = 0$. Then there is $K \in \mathbb{N}$ such that for each $k \geq K$, $m_f(z^k) = f(z^k)$. Since $f \in \mathcal{F}$, $z^k \underset{k\to\infty}{\longrightarrow} z$, then $m_f(z^k) \underset{k\to\infty}{\longrightarrow} f(z) = m_f(z)$.

Case 2: There is $t > 0$ such that $r = a_t s$. For each $k \in \mathbb{N}$, let $r^k_+ \equiv \max\{a_t s^k, r^k\}$ and $r^k_- \equiv \min\{a_t s^k, r^k\}$. Observe that $4^t(r^k_+ - a_t s^k) \underset{k\to\infty}{\longrightarrow} 0$ and $4^{t-1}(r^k_- - a_{t-1}s^k) \underset{k\to\infty}{\longrightarrow} s/4$. Since $s^k \underset{k\to\infty}{\longrightarrow} s$ and $r^k \underset{k\to\infty}{\longrightarrow} a_t s$, then there is $K \in \mathbb{N}$ such that for each $k \geq K$, $r_k \in]a_{t-1}s^k, a_{t+1}s^k[$, and thus,

$$m_f(r^k, s^k, E^k) = \left\{ \begin{array}{cl} a_t E^k + \frac{1}{4^t}f\left(4^t(r^k_+ - a_t s^k), s^k, E^k\right) & \text{if } r^k = r^k_+ \\ a_{t-1}E^k + \frac{1}{4^{t-1}}f\left(4^{t-1}(r^k_- - a_{t-1}s^k), s^k, E^k\right) & \text{otherwise.} \end{array} \right.$$

Since $f \in \mathcal{F}$ then $f\left(4^t(r^k_+ - a_t s^k), s^k, E^k\right) \underset{k\to\infty}{\longrightarrow} 0$ and $f\left(4^{t-1}(r^k_- - a_{t-1}s^k), s^k, E^k\right) \underset{k\to\infty}{\longrightarrow} E/4$. Thus, $m_f(z^k) \underset{k\to\infty}{\longrightarrow} a_t E = m_f(a_t s, s, E) = m_f(z)$.

Case 3: There is $t \in \mathbb{N}$ such that $r \in]a_t s, a_{t+1}s[$. Since $s^k \underset{k\to\infty}{\longrightarrow} s$ and $r^k \underset{k\to\infty}{\longrightarrow} a_t s$, then there is $K \in \mathbb{N}$ such that for each $k \geq K$, $r_k \in]a_t s^k, a_{t+1}s^k[$ and thus,

$$m_f(z^k) = a_t E^k + \frac{1}{4^t}f\left(4^t(r^k - a_t s^k), s^k, E^k\right).$$

Since $f \in \mathcal{F}$, then $m_f(z^k) \underset{k\to\infty}{\longrightarrow} a_t E + \frac{1}{4^t}f\left(4^t(r - a_t s), s, E\right) = m_f(z)$.

Case 4: $r = s/3$. For each $k \in \mathbb{N}$, let $t(k) \in \mathbb{N}$ be such that $r^k \in]a_{t(k)}s^k, a_{t(k)+1}s^k]$. Since $s^k \underset{k\to\infty}{\longrightarrow} s$ and $r^k \underset{k\to\infty}{\longrightarrow} s/3$, then $t(k) \underset{k\to\infty}{\longrightarrow} \infty$. Now, since f is continuous, then $f(\cdot, s, E)$ is bounded. Thus, $m_f(z^k) \underset{k\to\infty}{\longrightarrow} \lim_{k\to\infty} a_{t(k)}E^k = E/3 = m_f(z)$.

• **(P1).** Let $(s, E) \in \mathbb{R}^2_+$ be such that $s \geq E$. Let $\{r, r'\} \subset [0, s/3]$ be such that $r' \geq r$. We prove that $m_f(r', s, E) \geq m_f(r, s, E)$. There are four cases.

Case 1: $r = 0$. Since $f \geq 0$, then $m_f(r', s, E) \geq 0 = m_f(r, s, E)$.

Case 2: There is $t \in \{0\} \cup \mathbb{N}$ such that $\{r, r'\} \subset]a_t s, a_{t+1}s]$. Since $r' \geq r$, then $4^t(r' - a_t s) \geq$

$4^t(r-a_ts)$. Since $f\in\mathcal{F}$, then $f(\cdot,s,E):[0,s/4]\to[0,E/4]$ is non-decreasing. Thus,

$$m_f(r',s,E)-m_f(r,s,E)=\frac{1}{4^t}\left[f\left(4^t(r'-a_ts),s,E\right)-f\left(4^t(r-a_ts),s,E\right)\right]\geq 0.$$

Case 3: There are $\{t,t'\}\subset\{0\}\cup\mathbb{N}$ such that $t\neq t'$, $r\in]a_ts,a_{t+1}s]$ and $r'\in]a_{t'}s,a_{t'+1}s]$. Since $r'\geq r$, then $t'\geq t+1$. Now, since $f\in\mathcal{F}$, then $0\leq f(\cdot,s,E)\leq E/4$. Thus,

$$m_f(r,s,E)\leq a_tE+E/4^{t+1}=a_{t+1}E\leq a_{t'}E.$$

Consequently, $m_f(r,s,E)\leq a_{t'}E+f(4^{t'}(r-a_{t'}s),s,E)/4^{t'}=m_f(r',s,E)$.

Case 4: $r>0$ and $r'=s/3$. Let $t\in\{0\}\cup\mathbb{N}$ be such that $r\in]a_ts,a_{t+1}s]$. Since $f\in\mathcal{F}$, then $0\leq f(\cdot,s,E)\leq E/4$. Thus, $m_f(r,s,E)\leq a_{t+1}E\leq E/3=m_f(r',s,E)$.

Finally, from the definition of m_f, $m(0,s,E)=0$ and $m(s/3,s,E)=E/3$.

- **(P2).** Let $(s,E)\in\mathbb{R}^2_+$ be such that $s\geq E$. Let $\{r,r'\}\subset[0,s/3]$ be such that $r'\geq r$. We prove that $m_f(r',s,E)-m_f(r,s,E)\leq r'-r$. There are four cases.

Case 1: There is $t\in\{0\}\cup\mathbb{N}$ such that $\{r,r'\}\subset]a_ts,a_{t+1}s]$. Thus,

$$m_f(r',s,E)-m_f(r,s,E)=\frac{1}{4^t}\left[f\left(4^t(r'-a_ts),s,E\right)-f\left(4^t(r-a_ts),s,E\right)\right].$$

Since $f\in\mathcal{F}$, then $f\left(4^t(r'-a_ts),s,E\right)-f\left(4^t(r-a_ts),s,E\right)\leq 4^t(r'-r)$. Thus, $m_f(r',s,E)-m_f(r,s,E)\leq r'-r$.

Case 2: There are $\{t,t'\}\subset\{0\}\cup\mathbb{N}$ such that $r=a_ts$ and $r'=a_{t'}s$. Since $4^t(r-a_ts)=0$ and $4^{t'}(r'-a_{t'}s)=0$, then $m_f(r',s,E)-m_f(r,s,E)=a_{t'}E-a_tE\leq a_{t'}s-a_ts=r'-r$.

Case 3: There are $\{t,t'\}\subset\{0\}\cup\mathbb{N}$ such that $t\neq t'$, $r\in]a_ts,a_{t+1}s]$ and $r'\in]a_{t'}s,a_{t'+1}s]$. By Cases 1 and 2,

$$m_f(r',s,E)-m_f(r,s,E)=\left\{\begin{array}{l} m_f(r',s,E)-m_f(a_{t'}s,s,E)+\\ m_f(a_{t'}s,s,E)-m_f(a_ts,s,E)+\\ m_f(a_ts,s,E)-m_f(r,s,E)\end{array}\right\}\leq r'-a_{t'}+a_{t'}-a_t+a_t-r.$$

Thus, $m_f(r',s,E)-m_f(r,s,E)\leq r'-r$.

Case 4: $r=0$ or $r'=\frac{s}{3}$. Let $\{r^k\}_{k\in\mathbb{N}}$ be a sequence in $]0,\frac{r'+r}{2}[$ such that $r^k\underset{k\to\infty}{\longrightarrow}r$, and let $\{v^k\}_{k\in\mathbb{N}}$ be a sequence in $]\frac{r'+r}{2},\frac{s}{3}[$ such that $v^k\underset{k\to\infty}{\longrightarrow}r'$. By Cases 1 to 3, for each $k\in\mathbb{N}$, $m_f(v^k,s,E)-m_f(r^k,s,E)\leq v^k-r^k$. Since m_f is continuous, then $m_f(r',s,E)-m_f(r,s,E)\leq r'-r$.

- **(P3).** Let $(s,E)\in\mathbb{R}^2_+$ be such that $s\geq E$. We prove that for each $r\in[0,s/3]$,

$$m_f\left(\frac{s+r}{4},s,E\right)=\frac{E}{4}+\frac{1}{4}m_f\left(r,s,E\right).$$

The function, $r'\in[s/4,s/3]\mapsto 4r'-s$, is a bijection from $[s/4,s/3]$ to $[0,s/3]$. Thus, the statement above is equivalent to: for each $r\in[s/4,s/3]$, $m_f(r,s,E)=\frac{E}{4}+\frac{1}{4}m_f(4r-s,s,E)$. We prove this last statement. Let $r\in[s/4,s/3]$. There are three cases.

Case 1: $r = s/4$. Since $f \in \mathcal{F}$, then $f(r,s,E) = E/4$. Since $m_f(4r-s,s,E) = 0$, then $m_f(r,s,E) = \frac{E}{4} + \frac{1}{4}m_f(4r-s,s,E)$.

Case 2: $r \in]s/4, s/3[$. Let $t \in \mathbb{N}$ be such that $r \in]a_t s, a_{t+1}s]$. Thus, $(4r-s) \in]a_{t-1}s, a_t s]$. Consequently,

$$m_f(4r-s,s,E) = a_{t-1}E + \frac{1}{4^{t-1}} f\left(4^{t-1}[(4r-s) - a_{t-1}s], s, E\right).$$

Since $m_f(r,s,E) = a_t E + \frac{1}{4^t} f\left(4^t(r - a_t s), s, E\right)$, then the direct calculation shows that $m_f(r,s,E) = \frac{E}{4} + \frac{1}{4}m_f(4r-s,s,E)$.

Case 3: $r = s/3$. Since $m_f(r,s,E) = E/3$ and $4r - s = r$, then

$$m_f(r,s,E) = \frac{E}{4} + \frac{1}{4}\left(\frac{E}{3}\right) = \frac{E}{4} + \frac{1}{4}m_f(4r-s,s,E).$$

$\square$

Proof of Proposition 3. Let S be a *continuous* rule that satisfies the *core axioms*. By Proposition 2, there is $f \in \mathcal{F}$ such that $S = S^f$. Let $(c,E) \in \mathcal{C}$. We prove that $CEA^*(c,E) \succeq_L S^f(c,E)$. Since $f \in \mathcal{F}$, then for each $r \in [0, \bar{c}/4]$, $f(r,\bar{c},E) \leq r$ and $f(r,\bar{c},E) \leq E/4$. Thus, $f \leq f_{CEA}$ and $m_f \leq m_{f_{CEA}}$. If $\min c = \max c$, then $CEA^*(c,E) = S^f(c,E)$. Thus, $CEA^*(c,E) \succeq_L S^f(c,E)$. If $\min c < \max c$, then $S^f_{[1]}(c,E) = m_f(\min c, \bar{c}, E) \leq m_{f_{CEA}}(\min c, \bar{c}, E) = CEA^*_{[1]}(c,E)$, and

$$S^f_{[3]}(c,E) = E - 2m_f\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right) \geq E - 2m_{f_{CEA}}\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right) = CEA^*_{[3]}(c,E).$$

Thus, $CEA^*(c,E) \succeq_L S^f(c,E)$.

To prove that CEA^* is the unique Lorenz dominant rule, let $f_* \in \mathcal{F}$ be such that for each $f \in \mathcal{F}$, $S^{f_*} \succeq_L S^f$. Let $(c,E) \in \mathcal{C}$. We prove that $S^{f_*}(c,E) = CEA^*(c,E)$. Note that since $f_* \in \mathcal{F}$, then for each $r \in [0, \bar{c}/4]$, $f_*(r,\bar{c},E) \leq r$ and $f_*(r,\bar{c},E) \leq E/4$. Thus, $f_* \leq f_{CEA}$ and $m_{f_*} \leq m_{f_{CEA}}$. If $\min c = \max c$, then $S^{f_*}(c,E) = CEA^*(c,E)$. If $\min c < \max c$, then $S^{f_*}_{[1]}(c,E) = m_{f_*}(\min c, \bar{c}, E) \leq m_{f_{CEA}}(\min c, \bar{c}, E) = CEA^*_{[1]}(c,E)$, and

$$S^{f_*}_{[3]}(c,E) = E - 2m_{f_*}\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right) \geq E - 2m_{f_{CEA}}\left(\frac{\bar{c} - \max c}{2}, \bar{c}, E\right) = CEA^*_{[3]}(c,E).$$

Since $f_{CEA} \in \mathcal{F}$, then $S^{f_*} \succeq_L CEA^*$. Then, $S^{f_*}_{[1]}(c,E) \geq CEA^*_{[1]}(c,E)$ and $S^{f_*}_{[3]}(c,E) \leq CEA^*_{[3]}(c,E)$. Altogether, $S^{f_*}_{[1]}(c,E) = CEA^*_{[1]}(c,E)$ and $S^{f_*}_{[3]}(c,E) = CEA^*_{[3]}(c,E)$. By Theorem 2, S^{f_*} and CEA^* satisfy *o-p-g*. Thus $S^{f_*}(c,E) = CEA^*(c,E)$. $\square$

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